EE 210 **Fall 2011**
Simultaneous Linear Equations **Dr. Blandford**

These notes explain how to solve linear simultaneous equations using Cramer's rule and the method of cofactors.

Definitions:

Minor – Given a square matrix $\overline{A}$ which has n rows and n columns and whose determinant is denoted by $|A|$, a minor of $\overline{A}$ is found by removing any row i and any column j and denoted by M_{ij}. The minor is itself another smaller determinant which has n-1 rows and n-1 columns.

Cofactor – A cofactor is a signed minor where the sign is determined by the following rule: $C_{ij} = -1^{(i+j)} M_{ij}$.

The determinant of any square matrix $\overline{A}$ can be written as the sum of cofactors along any row or column. In general, for an n x n matrix we can write

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 1:

Here we evaluate the determinant of $\overline{A}$ by evaluating its cofactors along the top row.

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = (1)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - (0)\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + (2)\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= (4x7 - 5x6) - 0 + 2x(3x6 - 4x5) = -6$$

Alternatively, we could evaluate the determinant using the second column (or any other row or column).

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = -(0)\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + (4)\begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} - (6)\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

$$= 0 + (4)(1x7 - 5x2) - 6x(1x5 - 3x2) = -12 - 30 + 36 = -6$$

Using the method of cofactors you can evaluate any sized determinant. This gets progressively more tedious as the size increases. For example a 4 x 4 breaks down into 4 cofactors each of which involves a 3 x 3 determinant. Each 3 x 3 determinant breaks down further into 3 cofactors.

Cramer's Rule:

Consider the circuit shown in Figure 1. There are three meshes in this circuit and we can write three mesh equations using KVL. These equations are:

$$m1: \quad 8 - 4(i_1 - i_3) - 2(i_1 - i_2) = 0$$

$$m2: \quad -2(i_2 - i_3) - 16 - 2(i_2 - i_1) = 0$$

$$m3: \quad -1(i_3) - 2(i_3 - i_2) - 4(i_3 - i_1) = 0$$

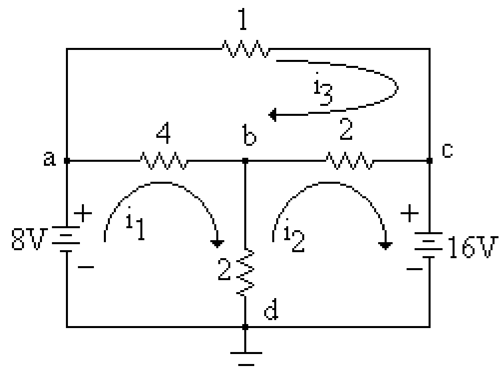

Figure 1

Using nodal analysis we get three equations with three unknowns.

In matrix form these equations become the following:

$$m1: \quad -6i_1 + 2i_2 + 4i_3 = -8$$
$$m2: \quad 2i_1 - 4i_2 + 2i_3 = 16 \qquad \text{or} \qquad \begin{bmatrix} -6 & 2 & 4 \\ 2 & -4 & 2 \\ 4 & 2 & -7 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 0 \end{bmatrix}$$
$$m3: \quad 4i_1 + 2i_2 - 7i_3 = 0$$

three new matrices by removing one column of the $\overline{A}$ matrix and replacing that column with the $\overline{b}$ matrix. Doing this leads to the following equations:

$$\overline{A_{i1}} = \begin{bmatrix} \mathbf{-8} & 2 & 4 \\ \mathbf{16} & -4 & 2 \\ \mathbf{0} & 2 & -7 \end{bmatrix} \qquad \overline{A_{i2}} = \begin{bmatrix} -6 & \mathbf{-8} & 4 \\ 2 & \mathbf{16} & 2 \\ 4 & \mathbf{0} & -7 \end{bmatrix} \qquad \overline{A_{i3}} = \begin{bmatrix} -6 & 2 & \mathbf{-8} \\ 2 & -4 & \mathbf{16} \\ 4 & 2 & \mathbf{0} \end{bmatrix}$$

Cramer's[1] rule states that we can find the values of the unknowns i_1, i_2, and i_3 from the following equations:

$$i_1 = \frac{|A_{i1}|}{|A|} \qquad i_2 = \frac{|A_{i2}|}{|A|} \qquad i_3 = \frac{|A_{i3}|}{|A|}$$

We can evaluate the required determinants using cofactors. In each case we will choose a row or column that contains zeros to minimize the multiplications of the cofactors.

$$|A_{i1}| = -8\begin{vmatrix} -4 & 2 \\ 2 & -7 \end{vmatrix} - 16\begin{vmatrix} 2 & 4 \\ 2 & -7 \end{vmatrix} + 0 = -8(24) - 16(-22) = 160$$

$$|A_{i2}| = -(-8)\begin{vmatrix} 2 & 2 \\ 4 & -7 \end{vmatrix} + 16\begin{vmatrix} -6 & 4 \\ 4 & -7 \end{vmatrix} - 0 = 8(-22) + 16(26) = 240$$

[1] Cramer's rule and the method of cofactors is derived in many books. For example see Ayres, Frank, Jr., Theory and Problems of Matrices, Schaum's Outline Series, McGraw-Hill, 1962.

$$|A_{i3}| = -8\begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix} - 16\begin{vmatrix} -6 & 2 \\ 4 & 2 \end{vmatrix} + 0 = -8(20) - 16(-20) = 160$$

$$|A| = -6\begin{vmatrix} -4 & 2 \\ 2 & -7 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 2 & -7 \end{vmatrix} + 4\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix} = -6(24) - 2(-22) + 4(20) = -20$$

These determinant values give

$i_1 = 160/-20 = -8$

$i_2 = 240/-20 = -12$

$i_3 = 160/-20 = -8$