

These notes explain how to solve linear simultaneous equations using Cramer's rule and the method of cofactors.

Definitions:

*Minor* – Given a square matrix  $\bar{A}$  which has n rows and n columns and whose determinant is denoted by  $|A|$ , a minor of  $\bar{A}$  is found by removing any row i and any column j and denoted by  $M_{ij}$ . The minor is itself another smaller determinant which has n-1 rows and n-1 columns.

*Cofactor* – A cofactor is a signed minor where the sign is determined by the following rule:  $C_{ij} = -1^{(i+j)} M_{ij}$ .

The determinant of any square matrix  $\bar{A}$  can be written as the sum of cofactors along any row or column. In general, for an n x n matrix we can write

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$

Example 1:

Here we evaluate the determinant of  $\bar{A}$  by evaluating its cofactors along the top row.

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = (1) \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - (0) \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + (2) \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= (4 \times 7 - 5 \times 6) - 0 + 2 \times (3 \times 6 - 4 \times 5) = -6$$

Alternatively, we could evaluate the determinant using the second column (or any other row or column).

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = -(0) \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + (4) \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} - (6) \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

$$= 0 + (4)(1 \times 7 - 5 \times 2) - 6 \times (1 \times 5 - 3 \times 2) = -12 - 30 + 36 = -6$$

Using the method of cofactors you can evaluate any sized determinant. This gets progressively more tedious as the size increases. For example a 4 x 4 breaks down into 4 cofactors each of which involves a 3 x 3 determinant. Each 3 x 3 determinant breaks down further into 3 cofactors.

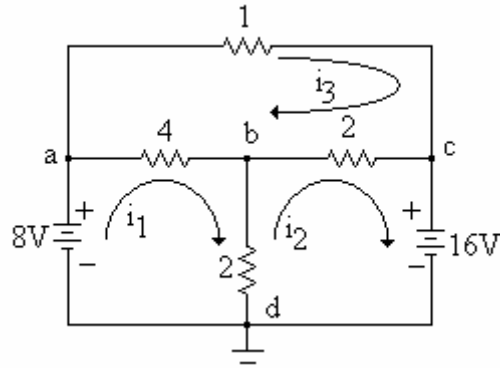
Cramer's Rule:

Consider the circuit shown in Figure 1. There are three meshes in this circuit and we can write three mesh equations using KVL. These equations are:

$$m1: \quad 8 - 4(i_1 - i_3) - 2(i_1 - i_2) = 0$$

$$m2: \quad -2(i_2 - i_3) - 16 - 2(i_2 - i_1) = 0$$

$$m3: \quad -1(i_3) - 2(i_3 - i_2) - 4(i_3 - i_1) = 0$$



**Figure 1**

Using nodal analysis we get three equations with three unknowns.

In matrix form these equations become the following:

$$\begin{aligned}
 m1: & -6i_1 + 2i_2 + 4i_3 = -8 \\
 m2: & 2i_1 - 4i_2 + 2i_3 = 16 \\
 m3: & 4i_1 + 2i_2 - 7i_3 = 0
 \end{aligned}
 \quad \text{or} \quad
 \begin{bmatrix} -6 & 2 & 4 \\ 2 & -4 & 2 \\ 4 & 2 & -7 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 0 \end{bmatrix}$$

three new matrices by removing one column of the  $\overline{A}$  matrix and replacing that column with the  $\overline{b}$  matrix. Doing this leads to the following equations:

$$\overline{A}_{i1} = \begin{bmatrix} -8 & 2 & 4 \\ 16 & -4 & 2 \\ 0 & 2 & -7 \end{bmatrix} \quad \overline{A}_{i2} = \begin{bmatrix} -6 & -8 & 4 \\ 2 & 16 & 2 \\ 4 & 0 & -7 \end{bmatrix} \quad \overline{A}_{i3} = \begin{bmatrix} -6 & 2 & -8 \\ 2 & -4 & 16 \\ 4 & 2 & 0 \end{bmatrix}$$

Cramer's<sup>1</sup> rule states that we can find the values of the unknowns  $i_1$ ,  $i_2$ , and  $i_3$  from the following equations:

$$i_1 = \frac{|A_{i1}|}{|A|} \quad i_2 = \frac{|A_{i2}|}{|A|} \quad i_3 = \frac{|A_{i3}|}{|A|}$$

We can evaluate the required determinants using cofactors. In each case we will choose a row or column that contains zeros to minimize the multiplications of the cofactors.

$$|A_{i1}| = -8 \begin{vmatrix} -4 & 2 \\ 2 & -7 \end{vmatrix} - 16 \begin{vmatrix} 2 & 4 \\ 2 & -7 \end{vmatrix} + 0 = -8(24) - 16(-22) = 160$$

$$|A_{i2}| = -(-8) \begin{vmatrix} 2 & 2 \\ 4 & -7 \end{vmatrix} + 16 \begin{vmatrix} -6 & 4 \\ 4 & -7 \end{vmatrix} - 0 = 8(-22) + 16(26) = 240$$

<sup>1</sup> Cramer's rule and the method of cofactors is derived in many books. For example see Ayres, Frank, Jr., Theory and Problems of Matrices, Schaum's Outline Series, McGraw-Hill, 1962.

$$|A_{i_3}| = -8 \begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix} - 16 \begin{vmatrix} -6 & 2 \\ 4 & 2 \end{vmatrix} + 0 = -8(20) - 16(-20) = 160$$

$$|A| = -6 \begin{vmatrix} -4 & 2 \\ 2 & -7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ 2 & -7 \end{vmatrix} + 4 \begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix} = -6(24) - 2(-22) + 4(20) = -20$$

These determinant values give

$$i_1 = 160 / -20 = -8$$

$$i_2 = 240 / -20 = -12$$

$$i_3 = 160 / -20 = -8$$