EE 311
Fast Fourier Transform

To show how the FFT can be done we take N to be a power of 2 and break the DFT up into two sequences of even and odd terms.

$$F(k) = \sum_{\substack{n=0 \\ n\ even}}^{N-1} f(n) e^{-j\frac{2\pi kn}{N}} + \sum_{\substack{n=0 \\ n\ odd}}^{N-1} f(n) e^{-j\frac{2\pi kn}{N}}$$

This equation can be simplified by doing a variable change of n to 2m.

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi k}{N}2m} + \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi k}{N}(2m+1)}$$

From this equation we can factor $e^{-j\frac{2\pi k}{N}}$ from the right most term to get

$$F(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi k}{N}2m} + e^{-j\frac{2\pi k}{N}} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi k}{N}2m} \qquad (1)$$

To simplify the notation let $G(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi k}{N}2m}$ and $H(k) = \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi k}{N}2m}$ so that equation (1) can be written as

$F(k) = G(k) + e^{-j\frac{2\pi k}{N}} H(k)$ where G(k) and H(k) are both N/2 term DFTs.

An N-term DFT is periodic with a period of N. Equation (1) is composed of two N/2-term DFT's and the period of each of these must be N/2. This implies that G(k) = G(k+N/2) and H(k) = H(k+N/2). To take advantage of this we write

$$F(k+N/2) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi(k+N/2)}{N}2m} + e^{-j\frac{2\pi(k+N/2)}{N}} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi(k+N/2)}{N}2m}$$

or, $F(k+N/2) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi k}{N}2m} \cdot e^{-j2m\pi} + e^{-j\frac{2\pi k}{N}} \cdot e^{-j\pi} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi k}{N}2m} \cdot e^{-j2m\pi}$

But $e^{-j2m\pi} = \cos(2m\pi) - j\sin(2m\pi) = 1$ since m is an integer and $e^{-j\pi} = -1$. This leads to

$$F(k+N/2) = \sum_{m=0}^{\frac{N}{2}-1} f(2m) e^{-j\frac{2\pi k}{N}2m} - e^{-j\frac{2\pi k}{N}} \sum_{m=0}^{\frac{N}{2}-1} f(2m+1) e^{-j\frac{2\pi k}{N}2m} \qquad \mathbf{(2)}$$

or $F(k+N/2) = G(k) - e^{-j\frac{2\pi k}{N}} H(k)$

Thus we see that if we calculate G(k) and H(k) to find F(k) in equation (1), we can make a single sign change and find F(k+N/2) using equation (2). To find all of the values of F(k) then, it is necessary to calculate all of the values of G(k) and H(k). This amounts to some considerable savings in computation time since G(k) and H(k) are N/2-term DFTs.

For example, if N = 16, a straight forward calculation of F(k) using the equation for the DFT would have N^2 = 256 complex operations. Calculation of F(k) from G(k) and H(k) would have only $2(N/2)^2 + N/2 = 136$ complex operations. (The N/2 term which is added in this equation is the result of the multiplication of $e^{-j\frac{2\pi k}{N}}$ time H(k)).

We can get further computational savings if we can repeat this process on G(k) and H(k). Thus, if N is a power of 2, N/2 is also a power of 2 and G(k) and H(k) can be divided into even and odd terms just as F(k) was. This division can continue until we are down to transforms of 1-term functions. Thus a 16 term function would be divided into two 8 point functions. These would be divided into four 4 point functions which would lead to eight 2 point functions. Each division would produce fewer complex operations. For an N-point sequence where $N = 2^p$, we can repeat this reduction process p times. Since $p = \log_2(N)$, the total number of complex multiplications is reduced to $(N/2)\log_2(N)$. ($N\log_2(N)$ additions are also required.) For example, if N = 1024, $\log_2(1024) = 10$ and 5,120 complex multiplications are required as opposed to 1,048,576 by brute force.

To visualize this consider the signal flow graph of Figure 1. In this figure, the central darkened circle is a summing junction and the signal flow graph shows how F(k) and F(k+N/2) are calculated from G(k) and H(k). Note that the ± sign on the e term must be chosen positive when calculating F(k) and negative when calculating F(k+N/2).

If an 8 point DFT is broken down into two 4 point DFTs, the butterfly signal flow graphs would be calculated as shown in Figure 2. In this figure the even terms of f(n) form the first 4 terms of G(k) and the odd terms of f(n) form the first four terms of H(k). If G(k) and H(k) are further broken down into even and odd terms the signal flow graph for F(k) is shown in Figure (3).

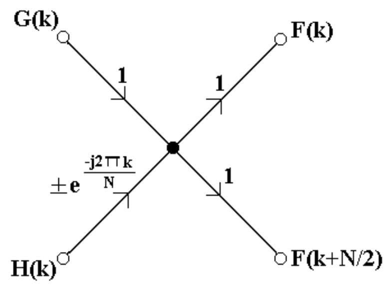

Figure 1

Signal flow graph for the calculation of F(k) and F(k+N/2) from G(k) and H(k) according to equations 1 and 2. This signal flow graph is commonly referred to a as a “Butterfly”.

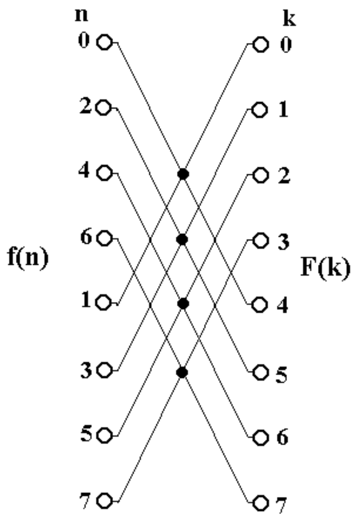

Figure 2
The butterfly signal flow graphs for the calculation of an 8-point FFT from two 4-point DFTs.

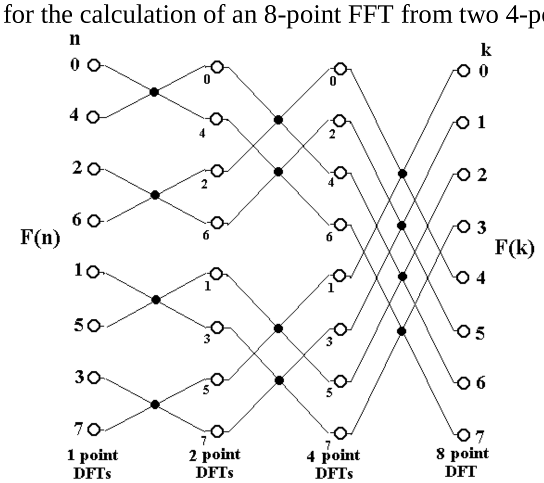

Figure 3
Butterfly signal flow graph for the calculation of an 8-point FFT from eight 1-point DFTs. The one point DFTs are the original function f(n).