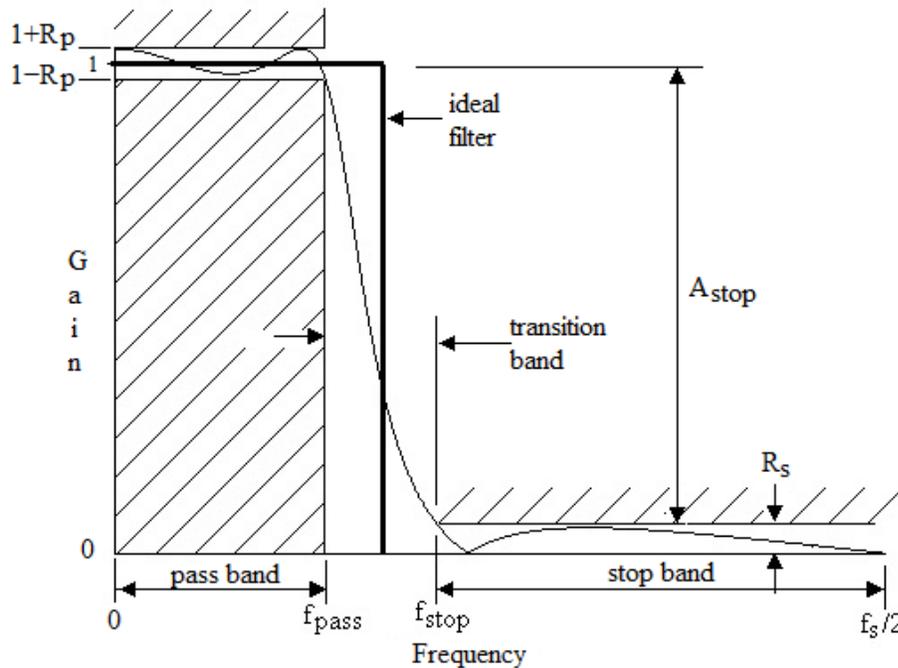




"I love the idea of there being two sexes, don't you?"

Chapter 5

The graph in Figure 5.1 is a *linear plot* but it is traditional to specify ripple and attenuation specification in decibels. The decibel is a unit for measuring the loudness of sound or sound pressure. In signal processing and communications applications, it is a measurement of signal power. The decibel is defined by the equation $db = 10 \log_{10}(P_{out} / P_{in})$. Since power is proportional to voltage squared the decibel can be written as $db = 20 \log_{10}(V_{out} / V_{in})$. The stop band attenuation is defined in decibels as $-20 \log_{10}(R_s)$ which is also referred to as the decibel ripple specification for the stop band. In the pass band the filter gain is normalized to one and the pass band ripple can go both above and below 1 as shown. The decibel pass band ripple is defined as $-20 \log_{10}(1-R_p)$.



Design using the Fourier series.

If the Fourier series is limited to $2N + 1$ terms, we will call this approximation $F_N(t)$ where

$$F_N(t) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cdot \cos(\omega_0 kt) + \sum_{k=1}^N b_k \cdot \sin(\omega_0 kt) = \sum_{k=-N}^N C_k \cdot e^{jk\omega_0 t}$$

From equations 3.5 and 3.14 we know that if a periodic function is approximated by a Fourier series for a finite number of terms, the *integral mean square error* is minimized for that number of terms.

To use this information in the design of an FIR filter recall that sampling in the time domain causes periodicity in the frequency domain and vice versa. An ideal low pass filter which is discrete in time will have a frequency characteristic similar to that shown in Figure 5.2.

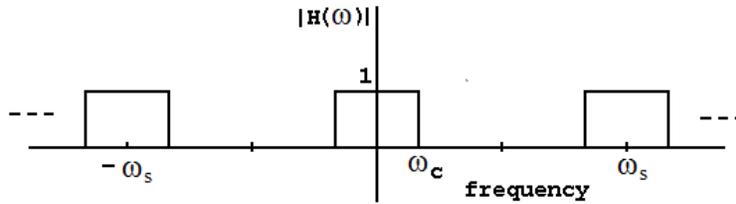


Figure 5.2

An ideal low pass filter that is sampled in the time domain.

If the signal in Figure 5.2 was in the time domain, we could find its Fourier series in the frequency domain. Since it already in the frequency domain we can do a change of variables:

Let $t \leftarrow f$ so that $T_0 \leftarrow f_s$

The exponential form of the Fourier series is:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad \text{where} \quad C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \quad \text{and} \quad \omega_0 = \frac{2\pi}{T_0}$$

When we do our variable transformation we get

$$H(\Omega) = \sum_{k=-\infty}^{\infty} C_k e^{jk2\pi f / f_s}$$

where

$$C_k = \frac{1}{f_s} \int_{f_s} H(\Omega) e^{-j2\pi k f / f_s} df.$$

Applying these equations to the signal of Figure 5.2 we get the following:

$$C_k = \frac{1}{f_s} \int_{-f_c}^{f_c} (1) e^{-j2\pi k f / f_s} df = \frac{1}{-2\pi k j} e^{-j2\pi k f / f_s} \Big|_{-f_c}^{f_c} = \frac{1}{k\pi} \sin(2\pi k f_c / f_s) \quad (5.1)$$

If the equation for $H(\Omega)$ is regarded as an evaluation of a transfer function $H(z)$ on the unit circle, we can replace $e^{j2\pi f / f_s} \leftarrow z$. This gives a transfer function in z with the problem that it is infinite in extent and noncausal.

$$H(z) = \sum_{k=-\infty}^{\infty} C_k z^{-k}$$

If we truncate this infinite series to $L = N+1$ terms and shift it in time by $N/2$ places we get the following causal, finite, transfer function in z . (N is the filter order and L is the filter length).

$$H(z) = \sum_{k=0}^N C_{k-N/2} z^{-k} \quad (5.2)$$

Example 5.2

Choose a sample frequency $f_s = 11,025\text{Hz}$, cutoff frequency $f_c = 2,000\text{Hz}$ and order $N = 10$ (length = 11) find the coefficients for a transfer function in z using the Fourier series method and the ideal filter characteristic of Figure 5.2.

Solution

From equation 5.1, $C_k = \frac{1}{k\pi} \text{Sin}(2\pi k f_c / f_s)$. Letting k go from -5 to +5 gives the following values for C_k

| k | C_k |
|-----|------------|
| -5 | -0.0351090 |
| -4 | -0.0786459 |
| -3 | -0.0291006 |
| -2 | 0.1208196 |
| -1 | 0.2892013 |
| 0 | 0.3628118 |
| 1 | 0.2892013 |
| 2 | 0.1208196 |
| 3 | -0.0291006 |
| 4 | -0.0786459 |
| 5 | -0.0351090 |

Table 5.1
Tabulated values of C_k using equation 5.1.

From equation 5.2 the transfer function for this FIR filter becomes:

$$H(z) = \frac{-0.03511z^{10} - 0.07865z^9 - 0.02910z^8 + \dots - 0.07865z - .03511}{z^{10}}$$

The corresponding magnitude and phase plots are shown in Figure 5.3.

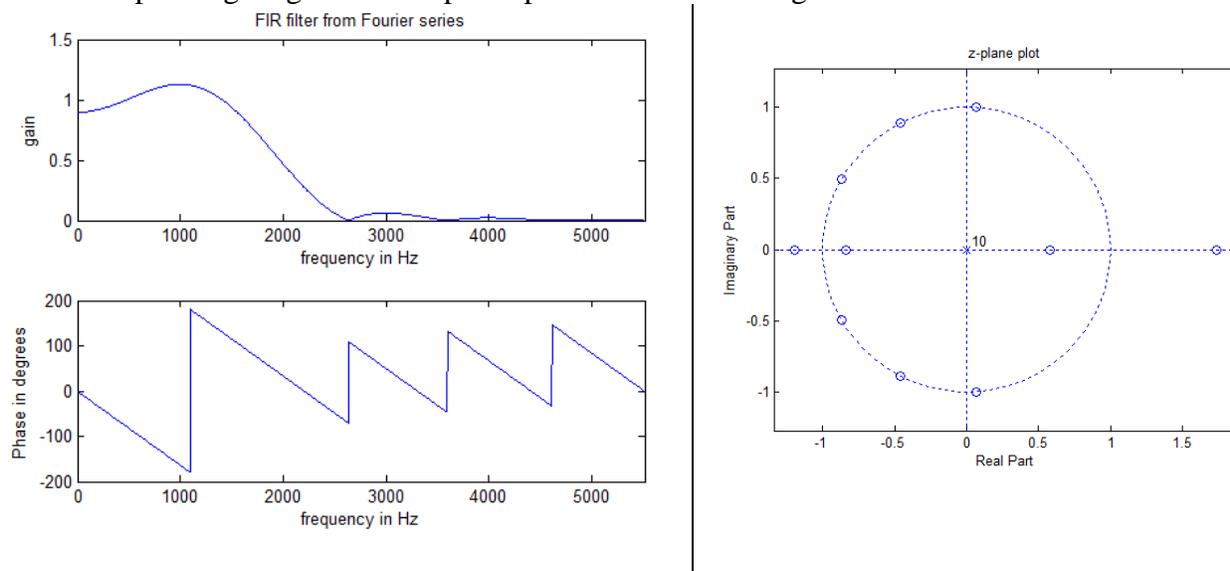


Figure 5.3
Magnitude and phase plots for the low pass FIR filter with a cutoff frequency at 2,000Hz.

The phase curve is said to be piecewise linear with the discontinuities after the first caused by zeros located directly on the unit circle.

The Fourier series design technique is implemented in MATLAB[®] as the function *firls*. The following lines of MATLAB[®] code produce a frequency response identical to that shown in Figure 5.3.

```

N = 10; %Order
fs = 11025;
F = [0 2000/(fs/2) 2000/(fs/2) 1]; %Frequency vector normalized to fs/2
M = [1 1 0 0]; %Corresponding magnitude vector
num = firls(N, F, M);
[H f] = freqz(num, 1, 1024, fs);
figure(1);
subplot(2, 1, 1)
plot(f, abs(H));
axis([0 fs/2 0 1.5]);
xlabel('frequency in Hz');
ylabel('gain');
title('FIR filter from Fourier series');
subplot(2, 1, 2);
plot(f, angle(H)*180/pi);
axis([0 fs/2 -200 200]);
xlabel('frequency in Hz');
ylabel('Phase in degrees');
figure(2);clf;
zplane(num, 1);
title('z-plane plot');

```

Linear phase and FIR filters

In general, FIR filters are computationally less efficient than IIR filters that meet the same specifications. However, FIR filters can be designed to have linear phase whereas IIR filters can only approximate linear phase and often, that is over a small band. Hence, for applications where information is encoded as part of the phase of a signal, having a filter that does not distort the phase curve is important.

We define two new terms related to phase:

$$\tau_p = -\frac{\theta(\omega)}{\omega} \equiv \text{Phase Delay} \quad (5.3)$$

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} \equiv \text{Group Delay} \quad (5.4)$$

The phase delay is simply the phase function normalized to the frequency. It represents the time delay in seconds that a particular sinusoid experiences as it passes through a system. For example, a single unit delay will have a transfer function of z^{-1} and a phase function of $\theta(\omega) = -\omega T$

The phase delay is

$$\tau_p = \omega T / \omega = T$$

and has units of seconds. In this case the phase delay is constant and all frequencies are delayed by T seconds.

Conditions for linear phase in FIR filters

For a filter with a constant phase delay and a constant group delay the phase response must be of the form:

$$\theta(\Omega) = -m\Omega \quad \text{where } \Omega = \omega T \quad (5.6)$$

This is the equation for a straight line with a slope of $-m$ giving us a linear phase response.

To determine what conditions produce a constant phase delay, constant group delay, and a linear response, we write the frequency response for an FIR filter from the transfer function.

$$H(z) = \sum_{n=0}^{L-1} h[n]z^{-n}$$

$$H(z)\Big|_{z=e^{j\Omega}} = \sum_{n=0}^{L-1} h[n]e^{-j\Omega n} = \sum_{n=0}^{L-1} h[n][\cos(\Omega n) - j \sin(\Omega n)]$$

From the frequency response we can write the phase response as the following:

$$\theta(\Omega) = \angle[H(e^{j\Omega})] = \tan^{-1} \left[\frac{-\sum_{n=0}^{L-1} h[n] \sin(\Omega n)}{\sum_{n=0}^{L-1} h[n] \cos(\Omega n)} \right] = -m\Omega \quad (5.7)$$

Since the tangent is the sine over the cosine we can rewrite this equation as:

$$\tan(\theta(\Omega)) = \frac{-\sum_{n=0}^{L-1} h[n] \sin(\Omega n)}{\sum_{n=0}^{L-1} h[n] \cos(\Omega n)} = \frac{\sin(-m\Omega)}{\cos(-m\Omega)}$$

If we cross multiply in this equation and put everything on the left side of the equals sign we get an equation which can be written as the sine of the sum of two angles.

$$\sum_{n=0}^{L-1} h[n] \sin(m\Omega) \cos \Omega n - \sum_{n=0}^{L-1} h[n] \sin \Omega n \cos(m\Omega) = 0$$

or, by applying the angle-difference relation of sines,

$$\sum_{n=0}^{L-1} h[n] \sin(m\Omega - \Omega n) = 0$$

The solution to this equation becomes more apparent if we expand it.

$$h[0] \sin(m\Omega) + h[1] \sin(m\Omega - \Omega) + h[2] \sin(m\Omega - 2\Omega) + \dots$$

$$+ h[L-2] \sin(m\Omega - \Omega[L-2]) + h[L-1] \sin(m\Omega - \Omega[L-1])$$

We see that we may satisfy this equation under the following conditions:

$$\sin(m\Omega) = -\sin(m\Omega - \Omega[L-1]) \quad \text{and} \quad h[0] = h[L-1]$$

$$\sin(m\Omega - \Omega) = -\sin(m\Omega - \Omega[L-2]) \quad \text{and} \quad h[1] = h[L-2]$$

$$\dots$$

$$\sin(m\Omega - n\Omega) = -\sin(m\Omega - \Omega[L-n-1]) \quad \text{and} \quad h[n] = h[L-n-1]$$

or

$$m\Omega - n\Omega = -m\Omega + \Omega(L - n - 1)$$

$$m = \frac{(L-1)}{2} \quad \text{and} \quad h[n] = h[L - n - 1] \quad (5.8)$$

Equation 5.8 gives conditions for having a linear phase response for an FIR filter. The equation for $h(n)$ stipulates that the impulse response must be symmetrical. There are two ways of achieving this symmetry depending on whether L is even or odd. Figure 5.5 shows typical impulse response functions of even and odd lengths with the two kinds of symmetry.

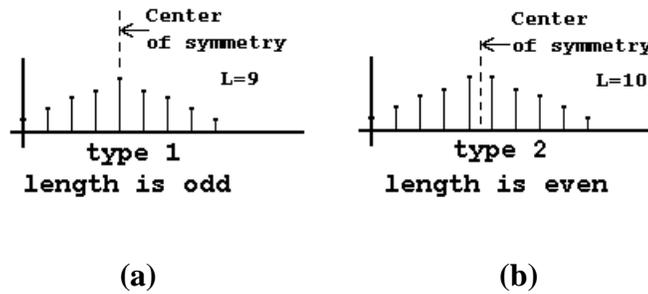


Figure 5.5

Typical impulse response functions with (a) odd and (b) even lengths. Both functions are symmetrical and therefore produce phase responses which are linear.

An FIR filter whose impulse response is symmetrical with an odd length (even order) as shown in Figure 5.5(a) is referred to as a *type 1* filter. If the length is even (odd order) the filter is referred to as a *type 2* filter.

In determining the conditions necessary for linear phase we had assumed that the phase response was of the form $\theta(\Omega) = -m\Omega$. This is the equation for a straight line that passes through the origin giving a constant phase delay, constant group delay, and a linear phase response. We could also write a straight line as $\theta(\Omega) = -m\Omega + \theta_0$ where θ_0 is a constant. If we use this equation the phase delay will no longer be a constant but will be a function of ω . The group delay however, will be constant. If we use this condition in equation 5.7 the solution can be calculated as the following (taking $\theta_0 = \pm\pi/2$):

$$m = \frac{(L-1)}{2} \quad \text{and} \quad h[n] = -h[L - n - 1] \quad (5.9)$$

The impulse response is said to be anti-symmetrical. Typical impulse response functions are shown in Figure 5.6 for the cases where the length is odd (type 3) and even (type 4).

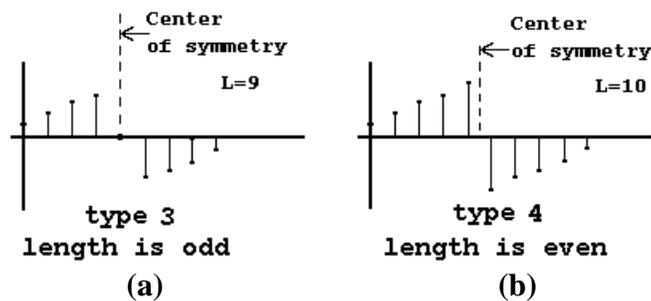


Figure 5.6

Typical impulse response functions with (a) odd and (b) even lengths. Both functions are anti-symmetrical and therefore produce phase responses which are linear.

For type 2 and type 4 FIR filters where the length of the filter is even, it is necessary to shift the filter in time by a fraction of a sampling period in order to get a causal filter. For example, In Figure 5.6(b), the center of symmetry falls between two samples so that the filter had to be shifted four and half samples to make it causal. It can be shown that the frequency response of type 1 and type 2 filters can be written as a sum of cosine functions while the response of type 3 and type 4 filters can be written as a sum of sine functions. The frequency response of the four filters therefore, places some restrictions on their usefulness. These restrictions are summarized in Table 5.2.

FIR filters of even length can be created by resampling the impulse response of an odd length filter in such a way as to achieve even symmetry. This is illustrated in Example 5.3.

| Type | Symmetry | Restrictions | Suitability |
|------|------------------------------|----------------------------|---|
| 1 | Odd length Symmetrical | None | Lowpass Highpass Bandpass Bandstop |
| 2 | Even length Symmetrical | $H(f_s/2)=0$ | Lowpass Bandpass |
| 3 | Odd length AntiSymmetric | $H(0)=0$ $H(f_s/2) = 0$ | Highpass Differentiator |
| 4 | Even length antiSymmetric | $H(0)=0$ | Highpass Differentiator |

Table 5.2

Restrictions and suitability for the various types of linear phase FIR filters.

Restrictions Imposed by Symmetry

To illustrate how symmetry in the impulse response limits the locations of the zeros, consider a specific case of a symmetric transfer function of length 5.

$$H(z) = K \frac{h[0]z^4 + h[1]z^3 + h[2]z^2 + h[3]z + h[4]}{z^4} = K \frac{N(z)}{D(z)}$$

If the impulse response is symmetric then $h[0] = h[4]$ and $h[1] = h[3]$. Using this information and factoring a z^2 out of the numerator gives the following:

$$N(z) = z^2[h[0]z^2 + h[1]z + h[2] + h[1]z^{-1} + h[0]z^{-2}]$$

In this equation we note that replacing z by its inverse would produce an equation which has the same roots.

$$N(z^{-1}) = z^{-2}[h[0]z^{-2} + h[1]z^{-1} + h[2] + h[1]z + h[0]z^2]$$

To generalize this for a filter whose length is L we write:

$$N(z) = z^{-m} \left(h[m] + \sum_{n=0}^{m-1} h[n](z^{-(m-n)} + z^{(m-n)}) \right) \text{ where } m = \frac{L-1}{2}$$

A similar result can be obtained for the cases where L is even or anti-symmetric.

Since replacing z by its inverse produces the same equation we can conclude that if there is a zero at a location z_1 then there must also be a zero at $1/z_1$. Combine this restriction on the zero locations along with the restrictions imposed by requiring that the transfer function have real coefficients and we arrive at the following conclusions about the zero locations.

Zero locations for FIR linear phase filters

1. Real zeros must appear with their inverse. This means that they will always appear in pairs except in the case where $z = \pm 1$ in which case the zero is its own inverse.
2. Complex zeros not on the unit circle must appear in sets of four (the zero plus its conjugate plus their inverses).
3. Complex zeros on the unit circle must appear as conjugate pairs. In this case, the conjugate(z) = inverse(z).

5.1 For each pole/zero plot in Figure P5.1 determine whether the corresponding filter is FIR or IIR, stable, causal, has linear phase, has minimum phase, and what type (LP, HP, BP, or BS).

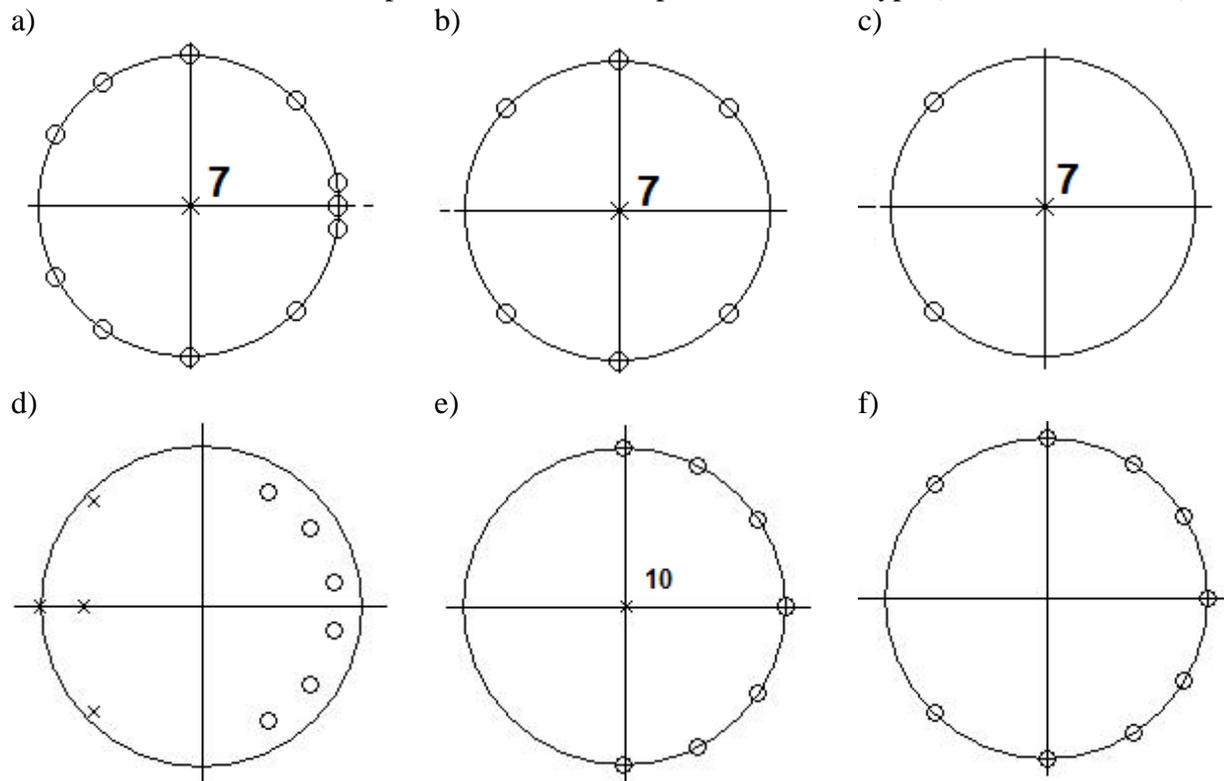


Figure P5.1
Pole/zero characteristics.

Window Functions and FIR Filters

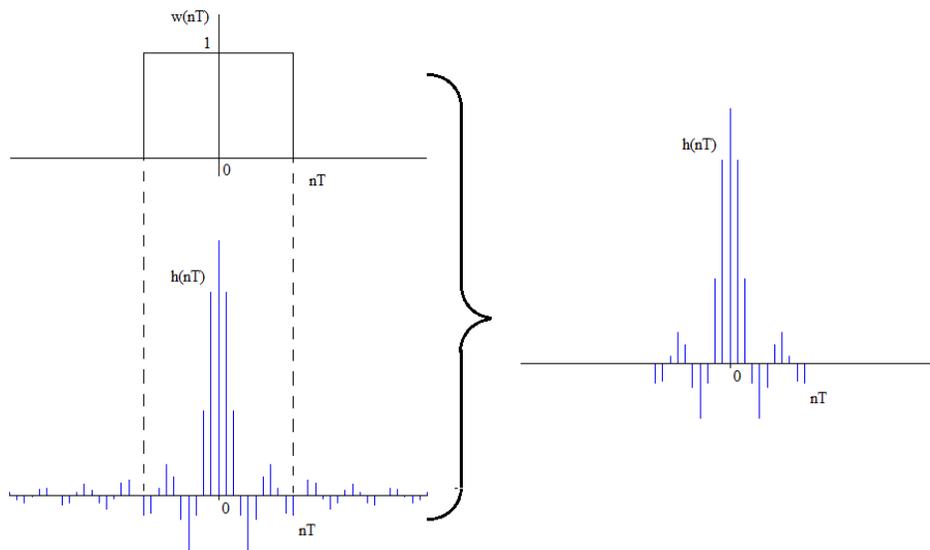


Figure 5.11

The window function (top) multiplies the impulse response function (bottom) to truncate the impulse response to a finite number of terms

Consider the process illustrated in Figure 5.12 using graphical convolution. We want to convolve the rectangular window with the ideal filter in the frequency domain. The equation for the convolution of the DFT's of two discrete signals is

$$Y(n) = H * W = \sum_{k=0}^{\infty} H(k) \cdot W(n - k)$$

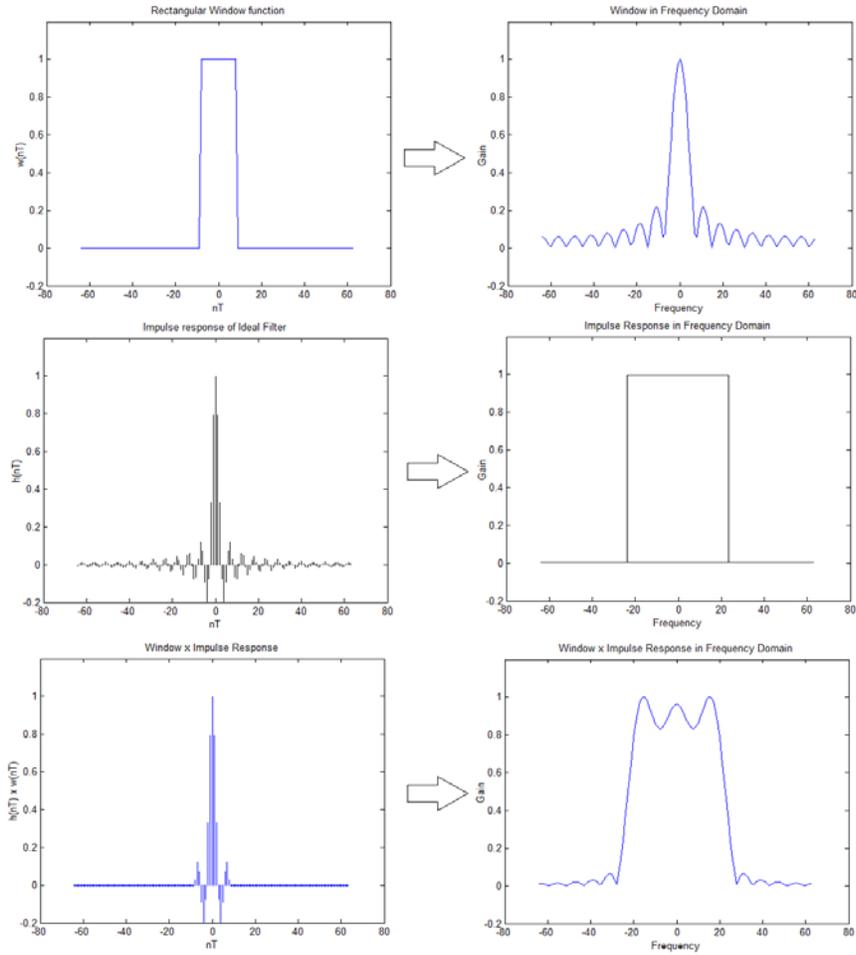


Figure 5.12

A rectangular window applied to an ideal low pass filter. Left column is the time domain and the right column is the equivalent frequency domain. The windowed filter in the frequency domain is the convolution of the ideal filter and the rectangular window. Note the ripple and the sloping transition band.

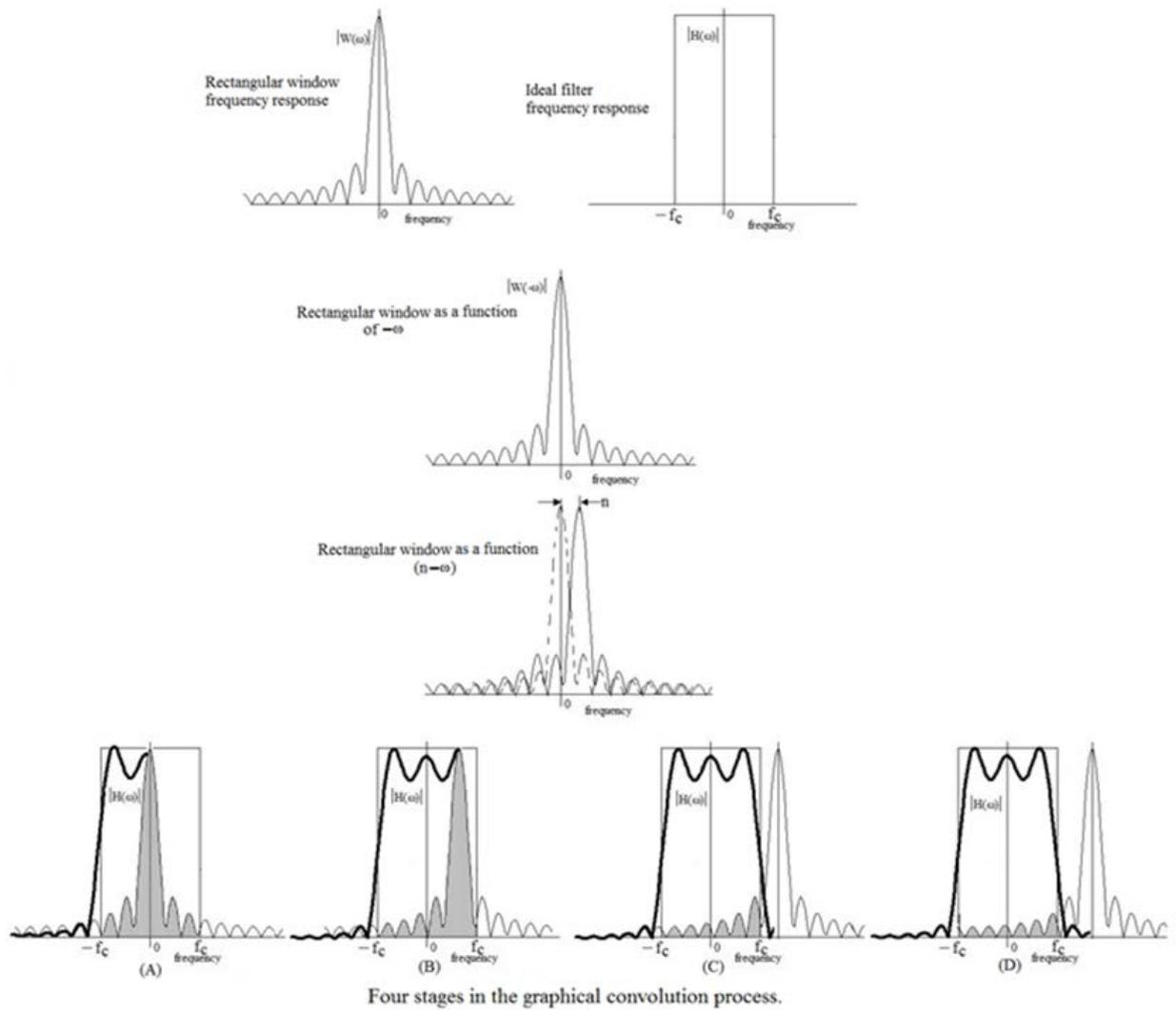


Figure 5.13

Graphical convolution of the impulse response of the ideal filter in the frequency domain with a rectangular window.

| Window Name | Function | MATLAB® |
|----------------------|--|---|
| Blackman | $w_B[n] = \begin{cases} .42 + .5 \cos\left(\frac{2\pi n}{L-1}\right) + .08 \cos\left(\frac{4\pi n}{L-1}\right) & n \leq (L-1)/2 \\ 0 & \text{otherwise} \end{cases}$ | <code>[w] = blackman(L);</code> |
| Hamming and von Hann | $w_H[n] = \begin{cases} \alpha + (1-\alpha) \cos\left(\frac{2\pi n}{L-1}\right) & n \leq (L-1)/2 \\ 0 & \text{otherwise} \end{cases}$ | <code>w = hamming(L);</code> <code>w = hann(L);</code> |

| | | |
|-------------------------|---|-----------------------------|
| | where $\alpha = .54$ for a Hamming window and $\alpha = .5$ for a von Hann window. | |
| Rectangular | $w_R[n] = \begin{cases} 1 & n \leq (L-1)/2 \\ 0 & \text{otherwise} \end{cases}$ | <code>w = boxcar(L);</code> |
| Triangular ¹ | $w_T[n] = \begin{cases} \frac{-2 n }{L+1} + 1 & n \leq (L-1)/2 \\ 0 & \text{otherwise} \end{cases}$ | <code>w = triang(L);</code> |

Note 1: There is another version of the triangular window called the "Bartlett" window which has slightly different end points. `w = bartlett(L);`

Table 5.4

Common window functions.

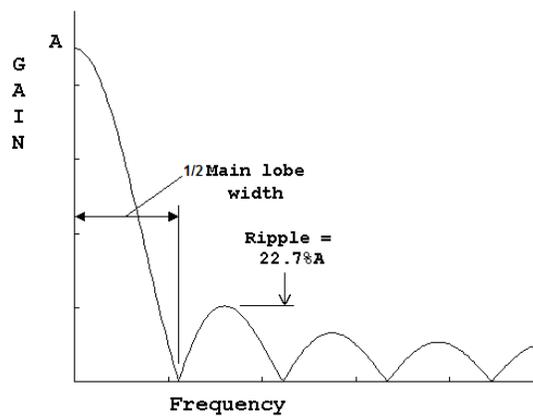


Figure 5.16

This is a rectangular window of length 9 in the frequency domain. Main lobe width is taken as the width to the first zero and the ripple is the maximum ripple.

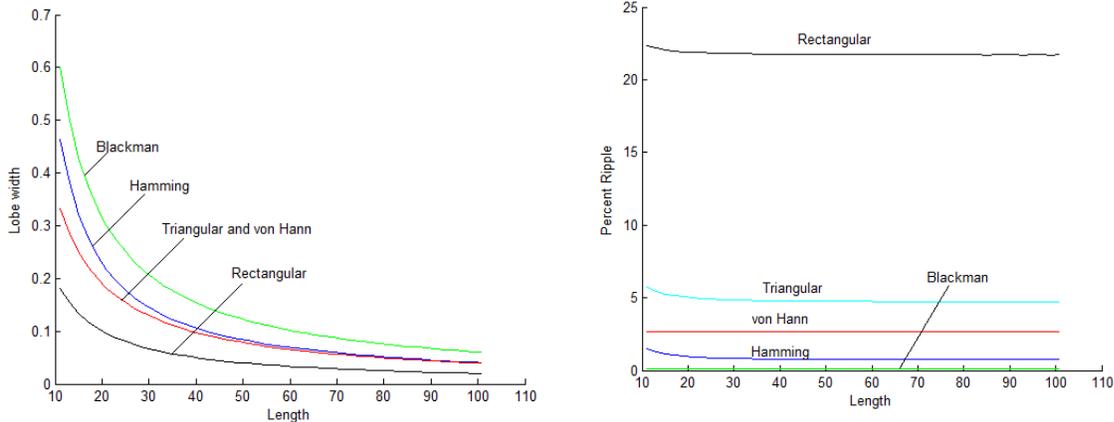


Figure 5.17

Variation in main lobe width and ripple with window length for odd-length windows. The main lobe width has been normalized to f_s .

EE 311
FIR Filter using Fourier Series
ANSWERS

February 15, 2012

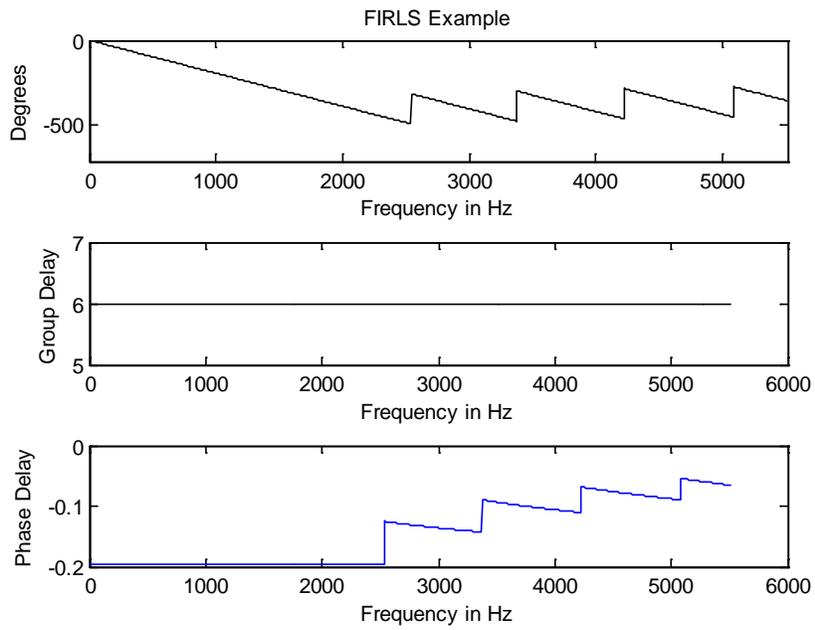
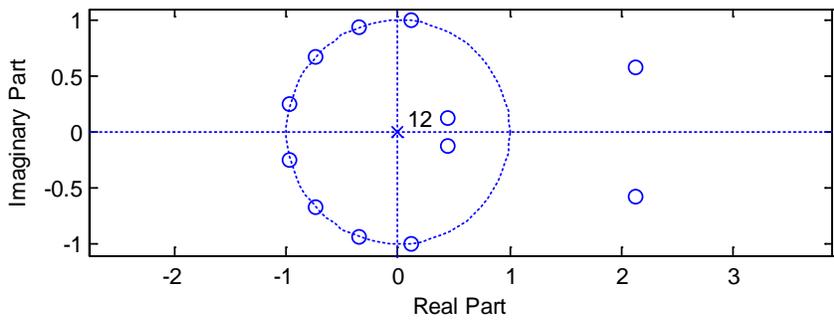
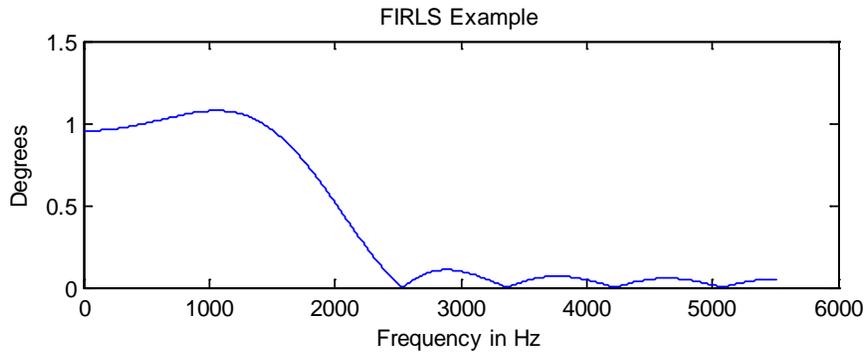
Use Matlab to design an FIR least squares filter that has the following ideal characteristics:

Sample frequency: 11025
pass band: 0 – 2000Hz
stop band: 2000Hz – fs/2
Order: 12

Plot the following:

Frequency vs gain
Frequency vs phase
pole/zero plot
Frequency vs phase delay
Frequency vs group delay

```
% FIRLSExmp.m
N = 12; fs = 11025;
F = [0 2000/(fs/2) 2000/(fs/2) 1];
M = [1 1 0 0];
num = firls(N, F, M);
den = 1;
[H f] = freqz(num, den, 1024, fs);
figure(1);clf;
subplot(2, 1, 1);
plot(f, abs(H));
ylabel('Degrees');xlabel('Frequency in Hz');
title('FIRLS Example');
subplot(2, 1, 2);
zplane(num, den);
figure(2);clf;
subplot(3, 1, 1);
HAngle = (180/pi)*unwrap(angle(H));
plot(f, HAngle, 'k');
axis([0 fs/2 -720 0]);
ylabel('Degrees');xlabel('Frequency in Hz');
title('FIRLS Example');
% Calculate the Group Delay
[Gd] = grpdelay(num,den,f,fs);
subplot(3,1,2)
plot(f,Gd,'k')
ylabel('Group Delay');xlabel('Frequency in Hz');
% Calculate the Phase Delay
subplot(3,1,3)
PhDelay = HAngle./(f);
plot(f, PhDelay)
ylabel('Phase Delay');xlabel('Frequency in Hz');
```



EE 311
FIR Windows in class
SOLUTION

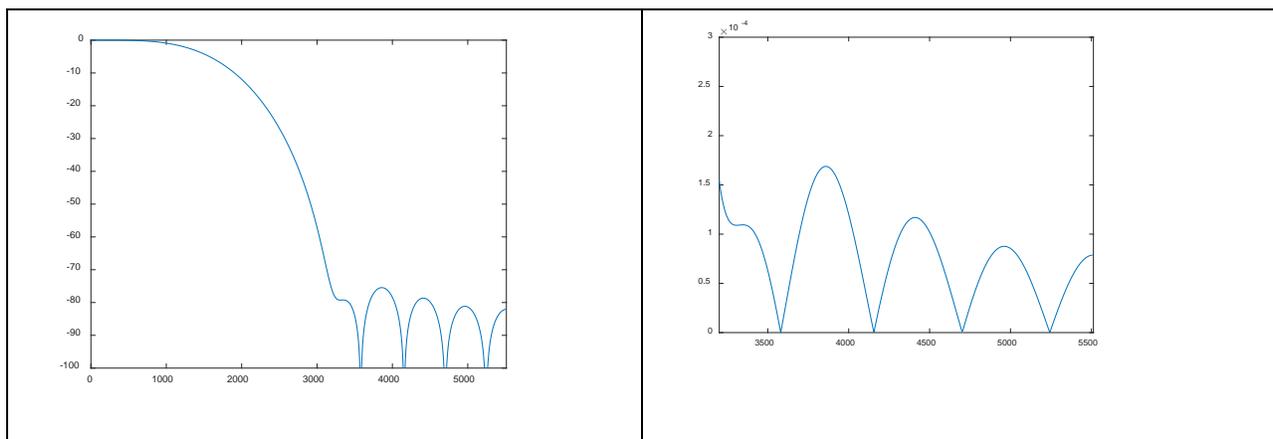
February 18, 2019

Design a windowed FIR filter of order no greater than 20 which meets the following specifications:

| | |
|--------------------------------|----------------|
| Stop band | 3200 to $fs/2$ |
| Cut off frequency (3 db point) | ≤ 1370 Hz |
| Stop band Ripple | -70db |
| Order | ≤ 20 |
| Sample Frequency | 11025 |

Use Matlab's axis statement to create blow-up diagrams to verify that your filter meets specifications.

```
%FIRWindowInClass.m
fs = 11025;
N = 20;
[num den] = fir1(N, .3, blackman(N+1));
[H f] = freqz(num, den, 1024, fs);
figure(1);clf;
db = 20*log10(abs(H));
figure(1);clf;
plot(f, db);
axis([0 11025/2 -100 0]);
figure(2);clf;
plot(f, abs(H));
axis([3200 fs/2 0 .0003]);
```



EE 311
Windowed Filter
SOLUTION

February 18, 2019

Design a low pass FIR filter with linear phase using a Hamming window to meet the following specifications

| | |
|------------------|---------|
| Sample frequency | 11025Hz |
| Pass band edge | 2000Hz |
| Pass band ripple | 0.02 |
| Pass band gain | 1.0 |
| Stop band edge | 3400Hz |
| Stop band ripple | 0.02 |
| Stop band gain | 0 |

```
%HammingWin.m
fs = 11025;
fps = 2000;
fst = 3400;
fc = (fps+fst)/2;
rp = .02;
rs = .02;
L = 23;N = L-1;
%Hamming
numH = fir1(N, fc/(fs/2), hamming(L));
[H f] = freqz(numH, 1, 1024, fs);
figure(1);clf;
plot(f, abs(H));
title('Hamming Filter');
xlabel('frequency in Hz');
ylabel('gain');
figure(2);clf;
subplot(2, 1, 1);
plot(f, abs(H));
title('pass band');
xlabel('frequency in Hz');
ylabel('gain');
axis([0 fps 1-rp 1+rp]);
subplot(2, 1, 2);
plot(f, abs(H));
title('stop band');
xlabel('frequency in Hz');
ylabel('gain');
axis([fst fs/2 0 rs]);
for i=1:length(numH);
    fprintf(1, '%3.12f\n', numH(i));
end
```

