

*"Is this man annoying you, dear?"*

**Summary of Classic IIR Filters**

While there are other types of digital IIR filters, the four classic filters (Butterworth, Chebyshev, inverse Chebyshev, and elliptic) presented here are the most popular. A summary of the characteristics of these filters is given in Table 6.4.

Filter Type	Pass Band	Transition Band	Stop Band	Comments
Butterworth	Maximally Flat	wide	Monotonic	Low pass has $N$ zeros at $f_s/2$ ., simple design procedure, order is relatively high to meet given specifications, stop band has more attenuation than is necessary
Chebyshev	Equiripple	medium	Monotonic	Low pass has $N$ zeros at $f_s/2$ , simple design procedure, order is higher than elliptic but lower than Butterworth for given specifications, stop band has more attenuation than is necessary
Inverse Chebyshev	Monotonic	medium	Equiripple	Design is slightly more complicated than Chebyshev, order is the same as Chebyshev
Elliptic	Equiripple	narrow	Equiripple	Design is mathematically complex, order is very low compared to Butterworth or Chebyshev for given specifications

**Table 6.4**  
Characteristics of classic filters.

**EE 311**  
**Digital Chebyshev Design**  
**SOLUTION**

**March 21, 2018**

1. Use Matlab to design a digital Chebyshev filter which meets the following specifications:

Sample frequency: 11025  
Pass band edge:  $fs/10$  Hz at 0.9  
Stop band edge:  $f_{pass} + fs/6$  Hz at 0.1

Find the numerator and denominator coefficients and create plots to demonstrate that your filter meets specifications.

```
%ChebyDesign.m
%Specifications: Digital filter
%sample frequency: 11025Hz
%Pass band 0 to fs/10 Hz
%Pass band ripple 0.1
%Stop band fpass + fs/6 Hz to infinity
%Stop band ripple 0.1
%*** SPECIFICATIONS ***
fs = 11025;
fpass = fs/10;
fstop = fpass + fs/6;
Rp = .1;RpDB = -20*log10(1-Rp);
Rs = .1;RsDB = -20*log10(Rs);
%***** Chebyshev Digital FILTER *****
[NC fn] = cheblord(fpass/(fs/2), fstop/(fs/2), RpDB, RsDB);
[num den] = cheby1(NC, RpDB, fn);
[H f] = freqz(num, den, 1024, fs);
figure(1);clf;
plot(f, abs(H), 'blue');
xlabel('frequency in Hz');
ylabel('Gain');
figure(2);clf;
subplot(2, 1, 1);
plot(f, abs(H), 'blue');
xlabel('frequency in Hz');
ylabel('Gain');
axis([0 fpass 1-Rp 1+Rp]); %Pass band blow up
subplot(2, 1, 2);
plot(f, abs(H));
axis([fstop fs/2 0 Rs]); %Stop band blow up
fprintf(1, 'Chebyshev order = %2i\n', NB);
fprintf(1, 'Numerator \n');
for i=1:NC+1
    fprintf(1, ' %2.12f \n', num(i));
end
fprintf(1, 'Denominator \n');
for i=1:NC+1
    fprintf(1, ' %2.12f \n', den(i));
end
```

## 6.6 Direct Design of IIR Filters

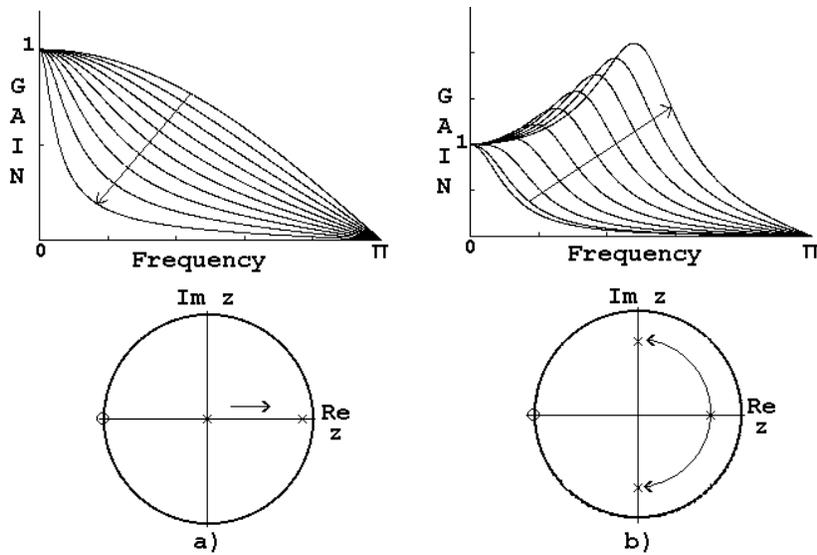
In the previous section we began the design of an IIR filter with an existing filter in the analog domain and used the BLT to transform that filter to the  $z$ -plane. It is also possible to design IIR filters without an analog filter counterpart. Such a design is referred to as "direct" design. With the appropriate software that provides nearly instant feedback, a designer can become adept at designing a low order IIR filter simply by placing poles and zeros on the  $z$ -plane to suit needs. Likewise, second order IIR filters that serve as notch filters and resonators can be designed with little more than a calculator. Finally, more advanced filters of higher order, can be designed using a computer to solve a set of simultaneous equations. The methods included here include Pade's method, Prony's method, and design using the Yule-Walker equations.

### Design by pole/zero placement

In section 6.3 we considered a geometric interpretation of the pole/zero plot to evaluate a transfer function. That is, in doing a magnitude or phase plot for a filter, we imagined a point moving along the unit circle in the  $z$ -plane with vectors to that point from each pole or zero in the transfer function. The gain for the filter was the product of the zero vectors divided by the product of the pole vectors. Using this interpretation, it is easy to visualize the effects of a single pole on the gain plot by looking at how the length of the vector from that pole to the point moving around the unit circle varies in length and phase. Clearly, a pole at the origin has no effect on the magnitude plot since the vector from the origin to the unit circle is always one. Poles close to the unit circle, on the other hand, have vectors which grow shorter as the evaluation point approaches them. Since it is necessary to divide by pole vector lengths, a pole close to the unit circle produces a high gain point at the frequency corresponding to the angle of the pole. Figure 6.28 illustrates this for a simple transfer function. This figure shows the gain plot and the corresponding pole movement for a real pole in part (a) and for a complex pole in part (b). In both cases the filter magnitude plot has been normalized to have a gain of unity at a frequency of zero. For a pole on the real axis the gain plot peaks at a frequency of zero. As the pole moves off the axis to become complex, the peak in the gain curve occurs at the frequency corresponding to the angle of the pole.

With Figure 6.28 in mind, and a geometric interpretation of the pole/zero plot, it is possible to place poles and zeros in the  $z$  plane to produce various kinds of filters. Poles and zeros must be placed in accordance with the following constraints.

1. Complex poles and zeros must appear as conjugate pairs. If this is not true, the filter coefficients will not be real.
2. The number of poles must be greater than or equal to the number of zeros. If this is not true, the filter will be noncausal.
3. No poles may be located outside the unit circle. This must be true for stability.
4. Poles may be placed at the origin to satisfy item 2. A pole at the origin has no effect on the magnitude plot.



**Figure 6.28**

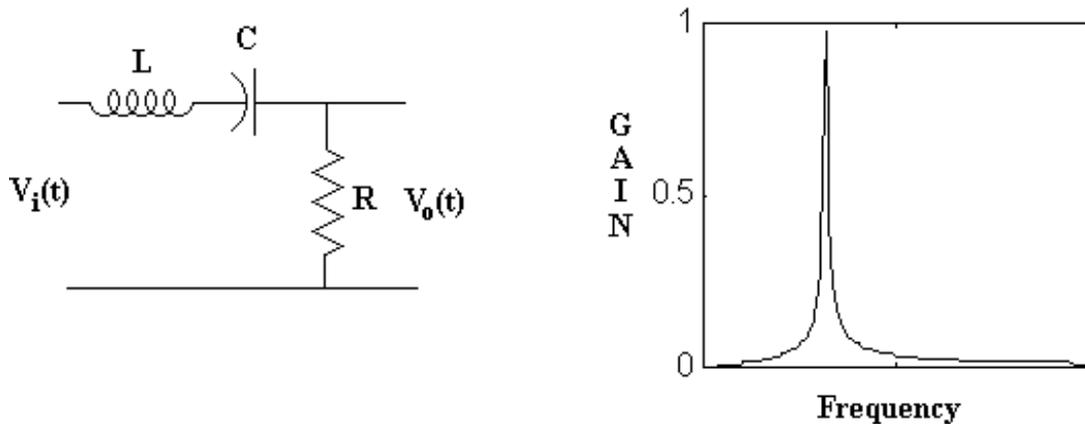
Pole movement and its relationship to the gain plot. In (a) the pole is real. In (b) the pole becomes complex. Note that the peak in the gain curve corresponds to the pole frequency.

This method is limited to low order filters since the interaction of a large number of pole/zero vectors is difficult to visualize. The method is very simple and easy to apply. To make this method effective a computer is required which can interactively produce a gain plot in response to a pole or zero movement. Problem 6.45 shows how to produce a MATLAB<sup>®</sup> program to make this process simpler.

### Design of resonators and notch filters of second order

A series RLC circuit such as that shown in Figure 6.29 is said to be "resonant" if the values of R, L, and C are chosen in such a way as to make the transfer function have complex roots. The transfer function is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{s(R/L)}{s^2 + s(R/L) + 1/(LC)} \quad (6.32)$$



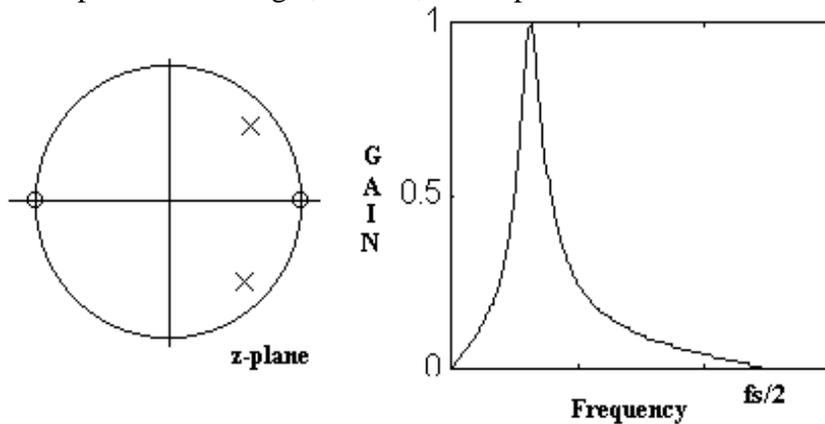
**Figure 6.29**

A resonant RLC circuit and its gain plot. For this plot  $R = 100\Omega$ ,  $L = 10\text{H}$ , and  $C = 1\mu\text{f}$ . The resonant frequency is at  $1/(LC)$  radians. The gain has been normalized to a peak of unity.

A digital resonator can be formed by applying the BLT to (6.32). In this case the transfer function in  $z$  has the form

$$H(z) = \frac{z^2 - 1}{z^2 + az + b} \quad (6.33)$$

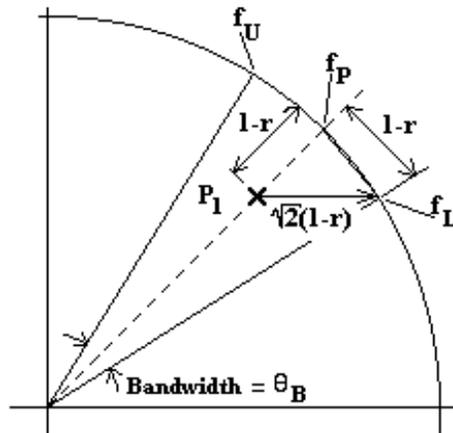
In this equation, if the values of  $a$  and  $b$  are chosen for complex roots, a typical pole/zero plot and a corresponding gain plot is shown in Figure 6.30. The BLT forces zeros in the  $z$ -plane at  $z = +1$  and  $z = -1$  so that the gain plot is exactly zero at the end points. The resonant peak in the gain plot corresponds to the angle,  $\theta = \omega T$ , of the poles.



**Figure 6.30**

A digital resonator pole/zero plot and corresponding gain plot.

Since this resonator has one complex pole pair and two real zeros, its design can be easily done by pole/zero placement. The bandwidth of a resonator can be defined as the size of the band of frequencies between the half power points. The half power points correspond to values where the normalized gain is  $\sqrt{2}/2$ . Recalling Section 6.3, we can evaluate the gain at a given frequency using vectors from each pole and zero to a point on the unit circle corresponding to a frequency to be evaluated. Figure 6.31 shows a geometric interpretation of the gain plot for a digital resonator.



### Figure 6.31

Geometric interpretation of an evaluation of the frequency response for a digital resonator.

In this figure, the point on the unit circle corresponding to the peak value is that point corresponding to the angle of the pole  $P_1$ . This point is labeled  $f_P$ . If a pole is located at  $r \angle \theta$ , then a vector from the pole to the unit circle at  $f_P$  is of magnitude  $1-r$ . Thus the poles contribution to the gain plot is  $1-r$  for the peak value of the gain plot. In Figure 6.31 the point  $f_L$  is the lower frequency half power point for the bandwidth and  $f_U$  is the upper frequency. If we assume that the contributions from the two real zeros and the complex conjugate pole remain approximately constant over the small frequency range from  $f_L$  to  $f_U$ , the pole at  $P_1$  must be responsible for the change in the gain over the band between  $f_L$  and  $f_U$ . As a consequence a vector from  $P_1$  to  $f_L$  must be of magnitude  $\sqrt{2}(1-r)$ . Thus we can write

$$\tan(\theta_B / 2) = 1 - r$$

For a resonator with a sharp peak, the angle  $\theta_B$  will be small and for small  $x$ ,  $\tan(x) \approx x$  so that

$$\theta_B \approx 2(1-r) = 2\pi f / f_s$$

From this we can write that

$$\text{Bandwidth} \approx \frac{f_s(1-r)}{\pi} \text{ Hz} \quad (6.34)$$

### Example 6.15

Design a digital resonator that has a sample frequency of  $f_s = 10,000\text{Hz}$  and a bandwidth of  $125\text{Hz}$  with a center frequency of  $1,000\text{Hz}$

#### Solution:

Putting values for the bandwidth and sample frequency in (6.34) gives

$$125 = \frac{10000(1-r)}{\pi}$$

Solving this equation gives  $r = 0.96073$  and  $r^2 = 0.92300$ .

The angle of the poles corresponds to the center frequency. The pole should be located at

$$\frac{2\pi(1000)}{10,000} = .62832 \text{ radians} = 36^\circ$$

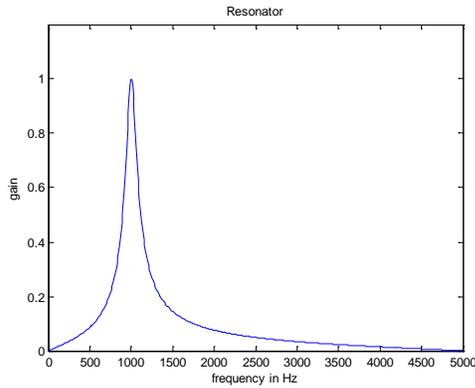
In Cartesian coordinates the pole is at  $.77725 \pm j.56470$ . The transfer function for the resonator is

$$H(z) = K \frac{z^2 - 1}{z^2 - 1.5545z + .92300}$$

where  $K$  is chosen to give a gain of unity at the peak.

The following lines in MATLAB<sup>®</sup> produce the gain plot shown in Figure 6.32.

```
fs = 10000;  
num = [1 0 -1];  
den = [1 -1.5545 .92300];  
[H f] = freqz(num,den,512,fs);  
biggest = max(abs(H));  
plot(f, abs(H)/biggest);
```



**Figure 6.32**  
Gain plot for a digital resonator.

Rearrangement of the RLC network of Figure 6.29 produces a notch filter as shown in Figure 6.33. The notch filter removes a narrow band of frequencies from its input. It is often used to remove noise at a known frequency such as power line noise at 60Hz.

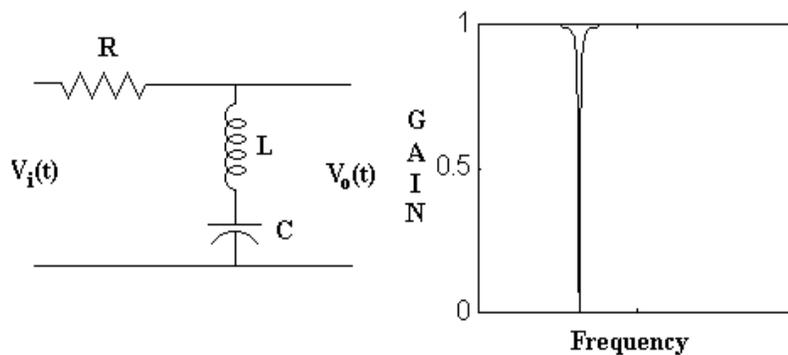
The transfer function for the RLC circuit is given by

$$H(s) = \frac{s^2 + 1/(LC)}{s^2 + (R/L)s + 1/(LC)}$$

Applying the BLT to this equation gives a transfer function in z in the form

$$H(z) = K \frac{z^2 + b_1z + b_2}{z^2 + a_1z + a_2}$$

where  $K$  is chosen to make the maximum gain unity. The coefficients in this equation are such that there is one set of complex zeros and one set of complex poles which are located at the same angular frequency. The zeros are located on the unit circle and the poles are just inside the circle. The distance that the pole is from the zero controls the bandwidth of the filter. It is easy to show that if the pole is located at  $r\angle\theta$ , the bandwidth can be approximated using the same equation as for the resonator.



**Figure 6.33**  
A series resonant RLC circuit which has a notch filter characteristic.

### Example 6.16

Design a filter to remove 60 Hz power line noise. The sample frequency is 1000 Hz and the bandwidth of the filter should be 4 Hz.

#### Solution:

For a notch filter the pole and zero are located at the same angle. In this case the angle is

$$\theta = \frac{2\pi f}{f_s} = \frac{2\pi(60)}{1000} = .37700 \text{ radians} = 21.60^\circ$$

The value of  $r$  for the pole can be calculated from (6.34).

$$\text{Bandwidth} \approx \frac{f_s(1-r)}{\pi} \text{ Hz}$$

or

$$4 \approx \frac{1000(1-r)}{\pi}$$

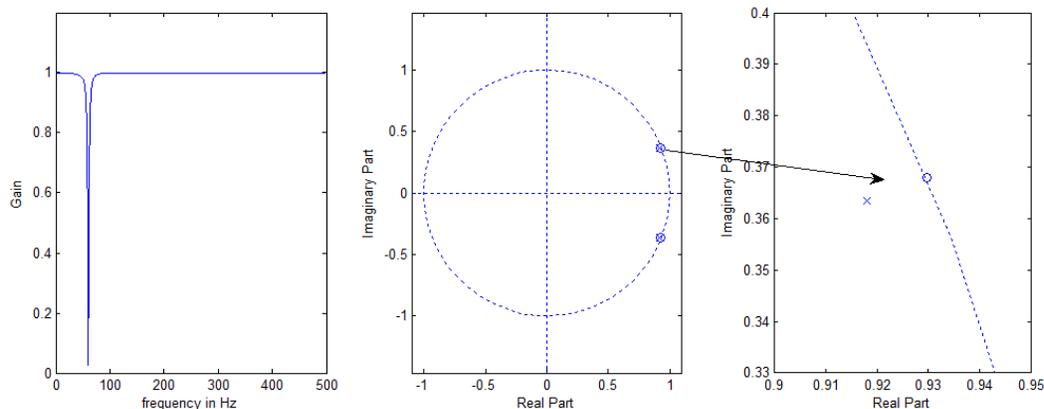
Solving this equation for  $r$  gives  $r = .98743$

The transfer function for the filter is

$$H(z) = K \frac{z^2 - 1.8596z + 1}{z^2 - 1.8362z + .97502}$$

The following lines in MATLAB<sup>®</sup> produce the plot shown in Figure 6.34.

```
fs = 1000;  
den = [1 -1.8362 .97502];  
num = [1 -1.8596 1];  
[H F] = freqz(num, den, 4096, fs);  
HMax = max(abs(H));  
H = abs(H)/HMax;  
plot(F,H);
```



**Figure 6.34**

The gain plot for a notch filter. The notch is centered at 60Hz and has a bandwidth of less than 4Hz. In the pole/zero plot at right, the poles are at  $0.98743\angle \pm 21.6^\circ$  and the zero is at  $1\angle \pm 21.6^\circ$ .