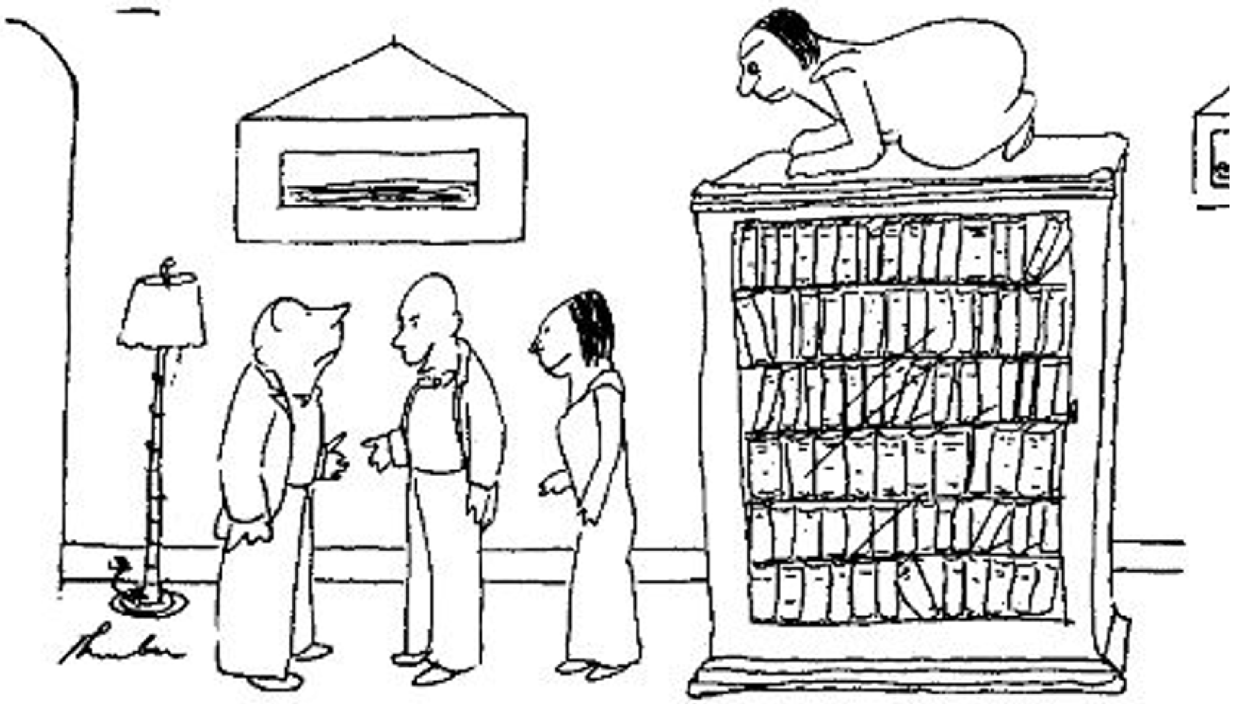


"That's my first wife up there, and this is the *present* Mrs. Harris."

EE 311 **January 23, 2019**
Notes – Fourth Day

Chapter 3

• Fourier series

The Fourier series uses the trigonometric functions as an orthogonal basis set. The trigonometric form of the Fourier series is given by the equation:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot Cos(\omega_0 kt) + \sum_{k=1}^{\infty} b_k \cdot Sin(\omega_0 kt) \qquad \textbf{2.4}$$

where

$$a_k = \frac{2}{T} \cdot \int_T f(t) \cdot Cos(\omega_0 kt)dt \quad \text{and} \quad b_k = \frac{2}{T} \cdot \int_T f(t) \cdot Sin(\omega_0 kt)dt$$

Example 2.1 Find the Fourier series for the square wave shown below.

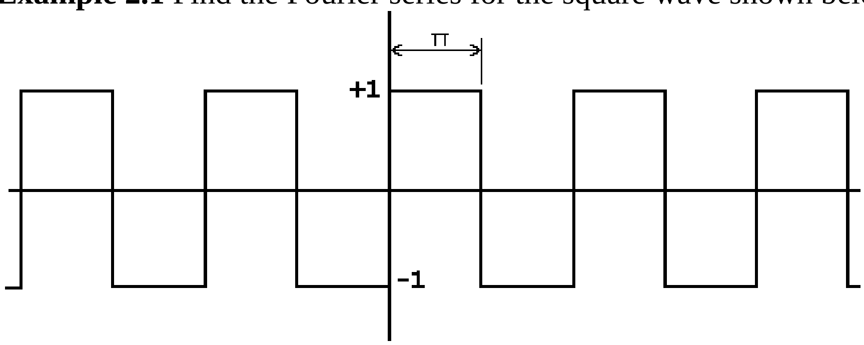


Figure E2.1-1
A square wave with a period of 2π.

Solution:

To find the coefficients a_k and b_k, we need to integrate over one period. For this problem we will take the period from 0 to 2π. The equation for f(t) is:

$$f(t) = \begin{cases} +1 & 0 \le t \le \pi \\ -1 & \pi \le t \le 2\pi \end{cases}$$

The equations for a_k and b_k can be evaluated as:

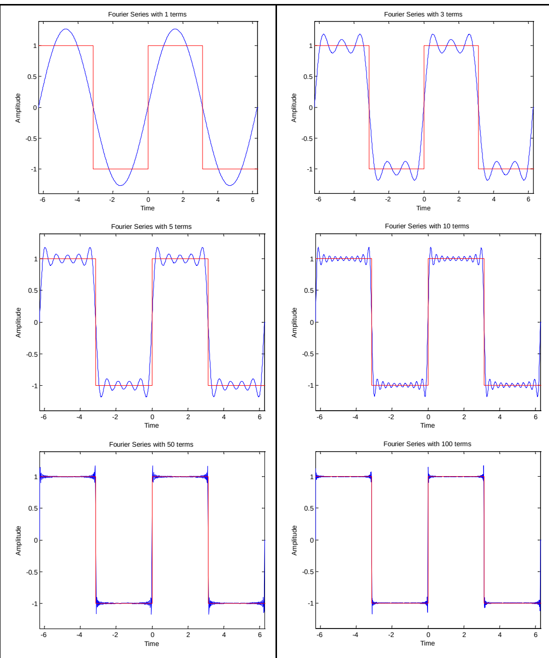
$$a_k = \frac{1}{\pi}\int_0^{\pi}(+1)Cos(kt)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)Cos(kt)dt \quad = 0$$

$$b_k = \frac{1}{\pi}\int_0^{\pi}(+1)Sin(kt)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)Sin(kt)dt \quad = \begin{cases} 0 & k \ even \\ \dfrac{4}{k\pi} & k \ odd \end{cases}$$

We can then write f(t) as a Fourier series.

$$f(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{k\pi} Sin(kt)$$

```
%Fourier.m
% This program plots terms of the
Fourier series for a square wave.
x1 = [-2*pi -pi -pi 0 0 pi pi 2*pi];
y1 = [1 1 -1 -1 1 1 -1 -1];
terms = [1 3 5 10 50 100];
%Vector of number of terms per plot
for indx = 1:6
  sum = zeros(1,1024);
  t = linspace(-2*pi,2*pi,1024);
  for k=1:2:2*terms(indx)
      y = 4/(k*pi)*sin(k*t);
      sum = sum + y;
  end
  figure(indx);clf;
  plot(t, sum, 'LineWidth', 1);
  hold on;
  plot(x1, y1, 'LineWidth', 1,
'Color','r');
  xlabel('Time');
  ylabel('Amplitude');
  sTitle = strcat(['Fourier Series
with ' int2str(terms(indx)) '
terms']);
  title(sTitle);
  axis([-2*pi 2*pi -1.4 1.4])
end
```



• Forms of the Fourier series

Using Euler's identity $e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$ we can change the trigonometric form of the Fourier series to the exponential form and other forms as shown in Table 3.2 in the text.

To change from the trigonometric form to the exponential form we start with

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot Cos(\omega_0 kt) + \sum_{k=1}^{\infty} b_k \cdot Sin(\omega_0 kt)$$

From Euler's identity we get

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

If we add these two equations we get

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Likewise, if we subtract these two equations we get

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Making these substitutions into the trigonometric form and collecting terms we get the exponential form of the Fourier series.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t};$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = |c_k|e^{j\theta_k}, \; c_{-k} = c_k^*$$

Note that in the trigonometric form of the Fourier series the counting variable k goes from 1 to ∞ whereas in the trigonometric form the counting variable goes from -∞ to +∞.

This change is the result of representing the sin and cos with $e^{-j\omega}$ terms. Effectively, this is where *negative* frequencies come from. Physically there is no such thing as a negative frequency but the mathematical representation in the exponential form introduces it.

Note also that the exponential form and the trigonometric form of the Fourier series both result in the same real coefficients. The –jω terms cancel with the +jω terms.

• Fourier series

Fourier Series Form	Equation
Exponential	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$; $c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt = \lvert c_k \rvert e^{j\theta_k}, \; c_{-k} = c_k^*$
Combined Trigonometric	$x(t) = c_0 + \sum_{k=1}^{\infty} 2\lvert c_k \rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + jb_k \sin(k\omega_0 t)],$ $a_0 = c_0 = \frac{1}{T_0}\int_{T_0} x(t) dt, \quad 2c_k = a_k - jb_k$ $a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t) dt$ and $b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t) dt$

The exponential form of the Fourier series is the most common and the easiest to work with.

The exponential Fourier series is:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

Notes

• f(t) is continuous in time but all frequencies are multiples of ω_0 and therefore discrete.

• The magnitude of $e^{-jk\omega t}$ is one so the magnitude of |f(t)| is given by the magnitude of c_k which is typically complex.

• In the frequency domain the Fourier series appears as a set of discrete frequency spikes at multiples of the fundamental frequency. Each spike has a magnitude of c_k.

For the square wave in the previous example which had a period of 2π and a 50% duty cycle the value of c_k can be determined from

$$c_k = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-1) e^{-jkt} dt + \int_{0}^{\pi} (+1) e^{-jkt} dt\right] \text{ where } \omega_0 = 2\pi / T_0 = 1$$

This evaluates to

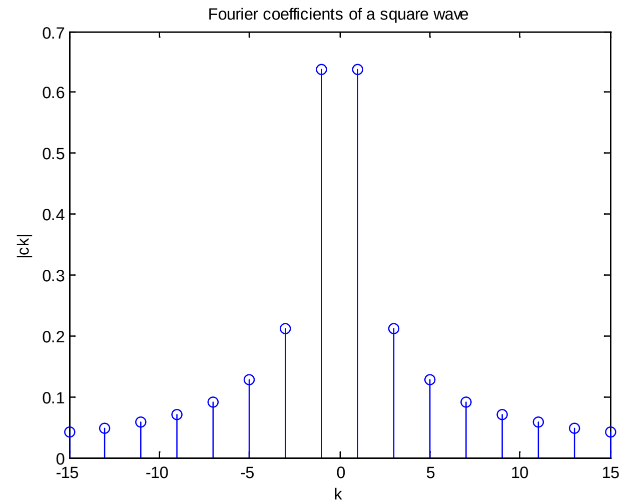
$$c_k = \begin{cases} 0 & \text{k is even} \\ 2/(jk\pi) & \text{k is odd} \end{cases}$$

(Recall that we got 4/(kπ) for the trigonometric form. In exponential form k is both positive and negative with $c_k = c_k^*$ and the magnitude of $c_k + c_k^*$ is still 4/(kπ).)

If we plot f(t) as magnitude vs frequency we need only plot $|c_k|$ vs frequency since the magnitude of the $e^{j\omega}$ term is one.

In MATLAB

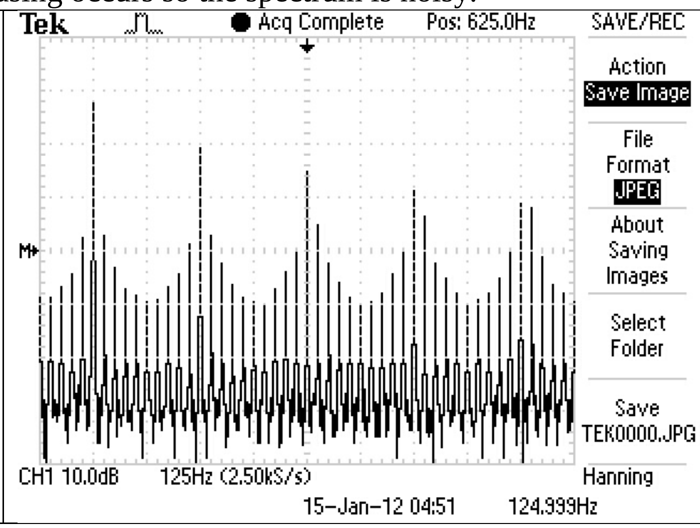
```
%SqWvFreqPlot.m
w0 = 1;
k = -15:2:15;
ck = abs(2./(k*pi));
figure(1);clf;
stem(k, ck);
title('Fourier coefficients of a square wave');
xlabel('k');
ylabel('|ck|');
```

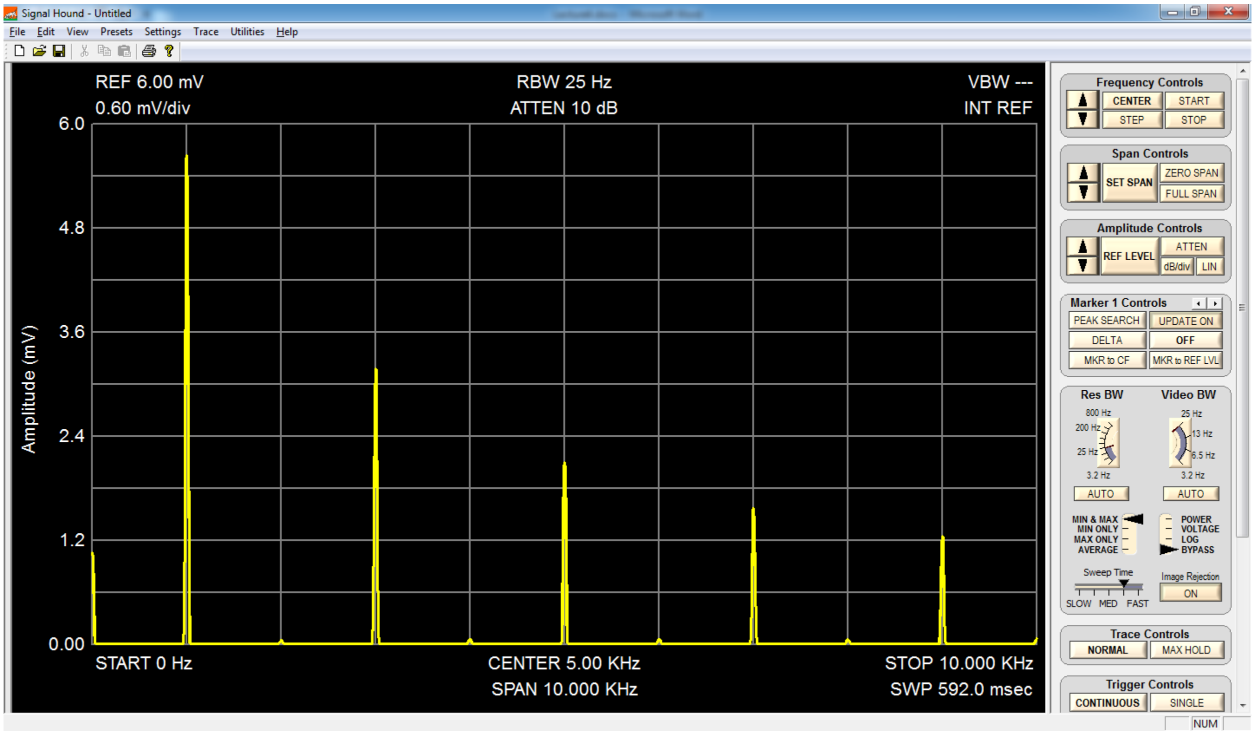


The x-axis is labeled 'k' but it is the frequency axis and is actually $k*\omega_0$ but $\omega_0 = 1$. We can do a similar graph with a spectrum analyzer.

This can be done on the Tectronics oscilloscope. The scope is not well suited for square waves since they contain high frequency and aliasing occurs so the spectrum is noisy.

Square wave is at 125 Hz with a 50% duty cycle. Note that vertical axis is in db with 10db per grid square.





This is a 1 KHz square wave with a 50% duty cycle.

Fourier series properties: These are listed in the text. Read these for yourself. In particular, property 4 will come up again with respect to FIR filter design.

4. We minimize the mean-square error, defined as

$$\text{mean-square error} = \frac{1}{T_0}\int_{T_0} e^2(t)\; dt. \tag{3.17}$$

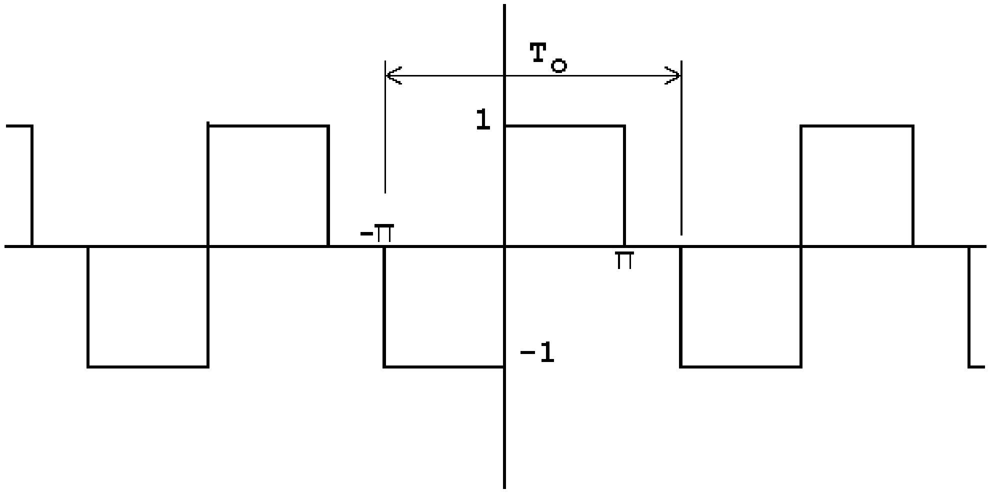
That is, *no other choice* of coefficients in the harmonic series (3.12) will produce a smaller mean-square error in (3.17).

• The Fourier Transform

The Fourier series is continuous in time and discrete in frequency. It allows us to write a *periodic* signal in terms of harmonics of its fundamental frequency.

All of the signals that we deal with in signal processing are not periodic and yet we need to represent them in the frequency domain. The Fourier transform does this – it represents a non-periodic signal in terms of its frequency components.

The Fourier transform can be derived from the Fourier series by creating a periodic function and allowing the period to go to infinity. In the figure below T_o is no longer 2π but the signal remains periodic. At T_o goes to infinity the signal approaches one which is not periodic.



The exponential Fourier series is given by

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} \quad \text{with} \quad c_k = \frac{1}{T_o}\int_{T_0} f(t)e^{-jk\omega_0 t}dt$$

The variable k is an integer and the increment in discrete frequency from one value of k to the next is $(k+1)\omega_0 - k\omega_0 = \Delta\omega = \omega_0$.

Since $\omega_0 = 2\pi/T_0$, we have

$$\Delta\omega = \lim_{T_0\to\infty}(2\pi / T_0) = d\omega$$

Also, the quantity $k\omega_0 = 2\pi k/T_0$ approaches $kd\omega$ as T_0 becomes infinite. Since *k* is infinitely variable over integer values, the product $kd\omega$ becomes the continuous frequency variable ω.

We can rewrite c_k as

$$c_k = \lim_{T_0\to\infty}\frac{1}{2\pi}\frac{2\pi}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-(jk2\pi/T_0)t/t} = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right]d\omega$$

The function in brackets is defined as the Fourier transform and can be written as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad \text{Fourier Transform}$$

$$c_k = \frac{1}{2\pi}F(\omega)d\omega$$

Putting this value of c_k into the equation for the exponential Fourier series gives

$$f(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} F(\omega)e^{jk\omega_0 t}d\omega$$

But with T_0 going to ∞ and ω_0 going to zero the term $k\omega_0$ goes to ω and the summation becomes an integral. This gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad \text{Inverse Fourier Transform}$$

Two ways to look at what the Fourier Transform is:
1. Consider c_k from the Fourier series

$$c_k = \frac{1}{T_o}\int_{T_0} f(t)e^{-jk\omega_0 t}dt$$

The Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

The Fourier transform can be considered to be equal to

$$F(\omega) = \lim_{T_0\to\infty} T_0 c_k$$

2. $F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Using Euler's identity we can replace $e^{-j\omega t}$ by $\cos(\omega t) - j\sin(\omega t)$ to get

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

If we do numerical integration by summing rectangular areas this equation becomes

$$F\{x(t)\} = X(\Omega) = T\sum_{n=-\infty}^{\infty} x(nT)\cos(\omega nT) - jT\sum_{n=-\infty}^{\infty} x(nT)\sin(\omega nT)$$

The cosine sum can be viewed as providing a sum which is a measure of the correlation between the signal $x(nT)$ and the cosine function at the frequency ω. Likewise, the sine sum provides a measure of the correlation between the signal and the sine function at frequency ω. The j term can be viewed as a marker to keep track of these two correlations. The following example illustrates this correlation view of the Fourier transform.

Example 11.1

Create a signal which has a value of $\cos(n\omega T - 30^o)$ for the first 50 msec, and a value of 0 for the last 50 msec. Apply (11.1) to this signal and show that it can be used to determine the magnitude and phase of the signal. Let the signal frequency be 100 Hz and the sample frequency be 1000 Hz.

Solution

In MATLAB® the following code creates the signal:

```
fs = 1000;T = 1/fs;
fsig = 100;
t = 0:T:.1-T;
x = cos(2*pi*fsig*t - pi/6);
x(length(t)/2:length(t)) = 0;
```

For any particular value of ω we can calculate the values for (11.1) as

```
sumCos = sum(cos(w*t).*x);
sumSin = sum(sin(w*t).*x);
mag = sqrt(sumCos^2 + sumSin^2);
phase = atan(-sumSin/sumcos)*180/pi;
```

These equations can be used to produce the magnitude plot below. The phase shift when $\omega = 2\pi(fsig)$ will be -30^o.

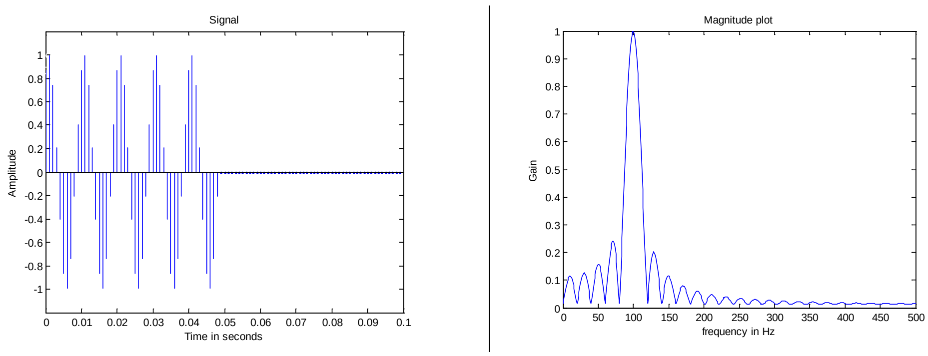


Figure 11.1
A cosine function (left) and the magnitude plot from equation (11.1).

There are two points of interest with regard to Example 11.1. First, we see clearly that the transformation from the time to the frequency domain can be viewed as a measure of the correlation between the signal and the sine and cosine functions which are the orthogonal basis functions for the Fourier transform. The second point of interest is what is *not* in the magnitude plot. There is no indication that the sinusoidal signal occurred only in the first half of the signal analyzed. All of the time information was lost.

• Properties of the Fourier transform

Operation	**Time Function**	**Fourier Transform**
linearity	$ax_1(t)+bx_2(t)$	$aX_1(\omega)+bX_2(\omega)$
time shift	$x(t-t_0)$	$X(\omega)e^{j\omega t_0}$
time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
time transformation	$x(at-t_0)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)e^{j\omega t_0/a}$
duality	$X(t)$	$2\pi x(-\omega)$
frequency shift	$x(t)e^{j\Omega_0 t}$	$X(\omega-\omega_0)$
convolution	$x_1(t)*x_2(t)$ $x_1(t)x_2(t)$	$X_1(\omega)X_2(\omega)$ $\frac{1}{2\pi}\left[X_1(\omega)*X_2(\omega)\right]$
time differentiation	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(\omega)$
frequency differentiation	$(-jt)^n x(t)$	$\frac{d^n X(\omega)}{d\omega^n}$
time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$