*"All right, have it your way—
you heard a seal bark!"*

**EE 311** **January 30, 2019**

**Notes – Seventh Day**

## Chapter 3

• *z*-transform properties

| PROPERTY | TRANSFORM PAIR | REGION OF CONVERGENCE |
|---|---|---|
| **Z-Transform notation** | $x[n] \overset{z}{\longleftrightarrow} X(z)$<br>$x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$<br>$x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$ | $R_x$<br>$R_1$<br>$R_2$ |
| **Linearity** | $ax_1[n]+bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z)+bX_2(z)$ | At least $R_1 \cap R_2$ |
| **Time Shift** | $x[n-k] \overset{z}{\longleftrightarrow} z^{-k}X(z)$ | $R_x$ with possible deletion or addition of the origin. |
| **Time Reversal** | $x[-n] \overset{z}{\longleftrightarrow} X(z^{-1})$ | $R_x^{-1}$ |
| **Frequency Shift** | $e^{j\Omega_0 n}x[n] \overset{z}{\longleftrightarrow} X(e^{-j\Omega_0 n}z)$ | $R_x$ |
| **Convolution** | $x_1[n]x_2[n] \overset{z}{\longleftrightarrow} X_1(z)*X_2(z)$ | At least $R_1 \cap R_2$ |
| **Differentiation in z** | $nx[n] \overset{z}{\longleftrightarrow} -z\dfrac{dX(z)}{dz}$ | $R_x$ |
| **Initial Value** | $x[0]=\lim_{z\to\infty} X(z),$ if $x[n]=0,\ n<0$ | |
| **Final Value** | $\lim_{n\to\infty} x[n]=\lim_{z\to 1}(z-1)X(z),$ iff $\lim_{n\to\infty} x[n]$ exists. | |

There is an error in the table on convolution:

If

$$y(n)=u(n)*x(n)$$

then

$$Y(z)=U(z)\cdot X(z)$$

We note that the converse of this property *does not apply*. That is, multiplication in time does not correspond to convolution in z.

The time shift property is probably the most important:

$$x(n-k) \leftrightarrow z^{-k}X(z)$$

Examples:

A) z-transform from difference equation

B) difference equation from transfer function in z

C) inverse z-transform by division

*Example 3.25 Inverse z-transform by long division*

The inverse transform in Example 3.18 is now verified by long division. From Example 3.22,

$$X(z)=\frac{z}{z-0.25}+\frac{z}{z-0.5}=\frac{2z^2-0.75z}{(z-0.25)(z-0.5)}=\frac{2z^2-0.75z}{z^2-0.75z+0.125}.$$

Dividing the numerator by the denominator yields

$$\begin{array}{r|l} & 2+0.75z^{-1}+0.3125z^{-2}\cdots \\ \hline z^2-0.75z+0.125 & 2z^2-0.75z \\ & \underline{2z^2-1.5z+0.25} \\ & \quad 0.75z-0.25 \\ & \quad \underline{0.75z-0.5625+0.09375z^{-1}} \\ & \qquad 0.3125-0.09375z^{-1} \\ & \qquad \vdots \end{array}$$

Hence, $x[0]=2.0$, $x[1]=0.75$, and $x[2]=0.3125$, which verifies solution found in Example 3.22. ■

We see that the power-series method is practical only for evaluating the first few values of a function unless the long division is implemented on a digital computer.

Alternatively,

$$X(z)=\frac{z}{z-0.25}+\frac{z}{z-0.5}; \quad |z|>0.5.$$

The poles and the ROC are plotted in Figure 3.23. Hence, $x[n]$ is right sided, and, from Table 3.9,

$$x[n]=\mathbf{Z}^{-1}[F(z)]=[0.25^n+0.5^n]u[n].$$

D) BIBO stability

First, we assume that $H(z)$ has no repeated poles. We can use partial fraction expansion to express the output $Y(z)$ in(3.86) as

$$\begin{aligned} Y(z)=H(z)\ X(z)&=\frac{b_0z^N+b_1z^{N-1}+\cdots+b_{N-1}z+b_N}{a_0(z-p_1)(z-p_2)\cdots(z-p_N)}X(z) \\ &=\frac{k_1z}{z-p_1}+\frac{k_2z}{z-p_2}+\cdots+\frac{k_Nz}{z-p_N}+Y_x(z), \end{aligned} \qquad (3.90)$$

where $Y_x(z)$ is the sum of the terms, in the partial-fraction expansion, that originate in the poles of the input function $X(z)$. Hence, $Y_x(z)$ is the *forced response*.

In the partial fraction expansion of (3.90), it is assumed that the order of the numerator of $H(z)$ is lower than that of the denominator. If the order of the numerator polynomial is equal to or greater than the order of the denominator polynomial, the partial-fraction expansion will include additional terms as in(3.87).

The inverse transform of(3.77) yields

$$y[n] = k_1 p_1^n + k_2 p_2^n + \cdots + k_N p_N^n + y_x[n] = y_n[n] + y_x[n]. \qquad (3.91)$$

The terms of $y_n[n]$ originate in the poles of the transfer function, and $y_n[n]$ is the *natural response*. The natural response is always present in the system output, independent of the form of the input signal $x[n]$. The factor $p_i^n$ in each term of the natural response is called a *mode* of the system.

If the input $x[n]$ is bounded, the forced response $y_x[n]$ will remain bounded, since $y_x[n]$ is of the functional form of $x[n]$. [$Y_x(z)$ has the same poles as $X(z)$.] Thus, an unbounded output must be the result of at least one of the natural-response terms, $k_i p_i^n$, becoming unbounded. This unboundedness can occur only if the magnitude of at least one pole, $|p_i|$, is greater than unity.

E) Frequency response – magnitude and phase.

# Second order systems

***Transfer function***

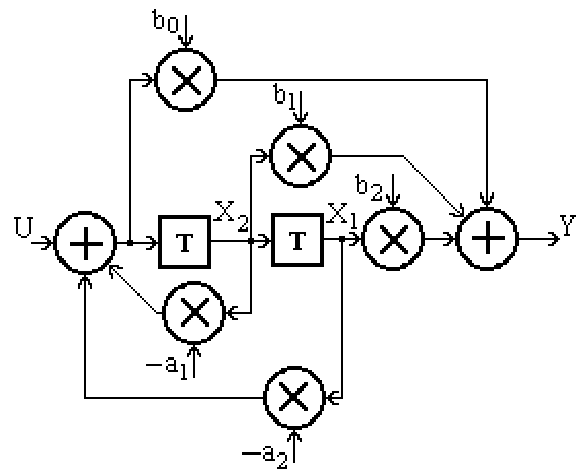
$$H(z) = \frac{y(z)}{u(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

***Difference equation***

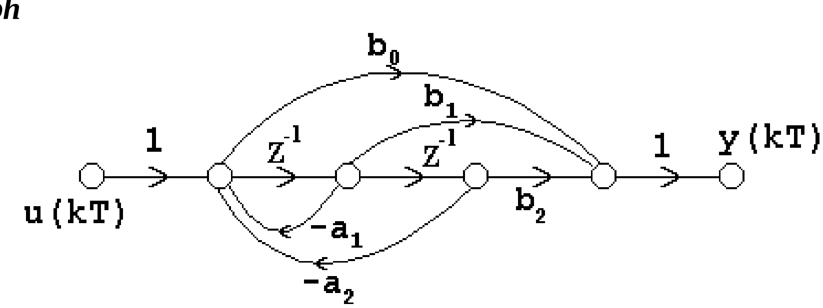
$$y_k = b_0 u_k + b_1 u_{k-1} + b_2 u_{k-2} - a_1 y_{k-1} - a_2 y_{k-2}$$

***State Variable Representation***

***Canonic Block diagram***




***Signal flow graph***




***State variable equations***

$x_1(k+1) = x_2$

$x_2(k+1) = -a_2 x_1 - a_1 x_2 + u_k \quad \Rightarrow \quad \overline{X(k+1)} = \overline{A} \cdot \overline{X(k)} + \overline{B} \cdot \overline{U(k)}$

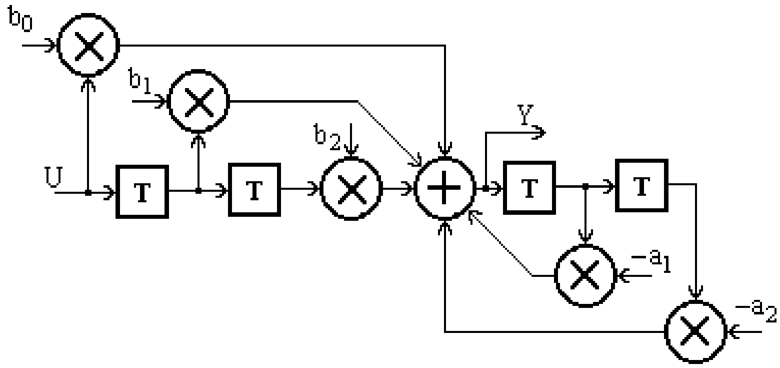
$y_k = (b_2 - b_0 a_2) x_1 + (b_1 - b_0 a_1) x_2 + b_0 u_k \quad \Rightarrow \quad \overline{Y(k)} = \overline{C} \cdot \overline{X(k)} + \overline{D} \cdot \overline{U(k)}$

where $\overline{X(k)} = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ $\quad \overline{Y(k)} = [y(k)]$ and

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \overline{C} = [b_2 - b_0 a_2 \; b_1 - b_0 a_1] \quad \overline{D} = [b_0]$$

***Direct form I***

***Block diagram***

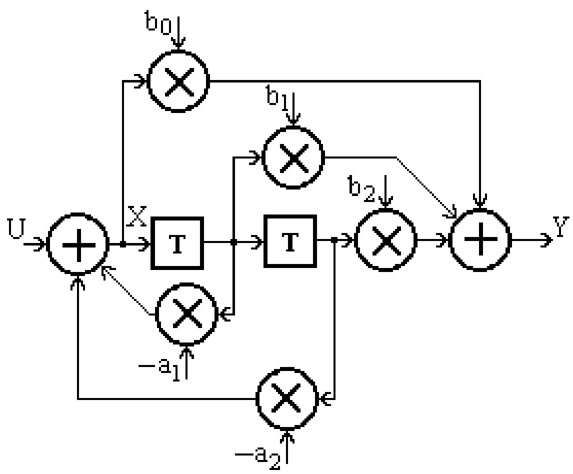



***Direct form I equations***

$$y_k = b_0 u_k + b_1 u_{k-1} + b_2 u_{k-2} - a_1 y_{k-1} - a_2 y_{k-2}$$

***Direct form II***

***Block diagram***




***Direct form II equations***

$$X(k) = U(k) - a_1 X(k-1) - a_2 X(k-2)$$
$$Y(k) = b_0 X(k) + b_1 X(k-1) + b_2 X(k-2)$$