MUPEC Design Competition Problem (with solution)

Surfaces of constant slope are geometries of great interest in several different branches of science and fields of engineering. For example, in civil engineering granular materials such as sand will form a surface of constant slope \( k \) (a material property) relative to a horizontal base when the granular material is poured onto the base under the influence of gravity. In the case of a circular plate, the sand will form a cone; provided sand has accumulated to the point where any additional sand will run off the side of the plate. In Figure 1(a) a representation of a right circular cone is drawn, whose generators are inclined at a constant slope \( k \) relative to the base.

![Figure 1](image)

**Figure 1** (a) Generators of a Right Circular Cone   (b) Height, Radius, and Slope

Mathematically a generator is a line whose motion traces out a class of surfaces known as ruled surfaces. The cone is just one example of a ruled surface. All surfaces of constant slope are also ruled surfaces. In fact, they belong to a special class of ruled surfaces called developable surfaces, which have additional metric properties that allow them to be constructed from a plane sheet of paper.

The volume of sand \( V \) resting on a circular plate of radius \( a \), which has been piled to a height \( h \), can be expressed mathematically by \( V = \frac{1}{3} \pi a^2 h \). Alternatively, this same volume could be expressed in terms of the slope \( k \) of its generators relative to its base as \( V = \frac{1}{3} \pi ka^3 \) - seeing from Figure 1(b) that \( k = h / a \) by similar triangles. When viewed from above, the generators of the cone in Figure 1(a) appear as radial lines emanating from the center of the circle. These projections of the generators onto the base are also normal to the circular boundary when viewed from this perspective.

In the case of an elliptical base piled high with sand, one would similarly find that the projections of the generators of this developable surface are normal to its plane.
boundary. In Figure 2, an elliptical boundary having a semimajor axis of length $a$ and a semiminor axis of length $b$ is shown.

Figure 2. Elliptical Boundary with Family of Parallel Curves and Several Normal Lines

Figure 3. Surface of Constant Slope having an Elliptical Base
This elliptical boundary may be expressed by the following algebraic formula

\[ \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1. \]  \hspace{1cm} (1.1)

Also portrayed in Figure 2 are some normal lines to the elliptical boundary and a family of parallel curves, i.e., curves that maintain a constant distance from the elliptical boundary. Collectively these normal lines and parallel curves form a natural orthogonal coordinate system in the xy plane.

A surface of constant slope having an elliptical base is shown in Figure 3. The generators of this ruled surface will appear as normal lines to the elliptical boundary when viewed from the xy plane as in Figure 2. Note a central ridge of sand with a discontinuous slope across the x axis forms along a portion of the x axis for the elliptical plate piled high with sand. As the eccentricity of the ellipse goes to zero, i.e., as the boundary becomes circular, this central ridge degenerates to a point.

The mathematical representation of any surface \( z = z(x, y) \) of constant slope \( k \) is

\[ \left( \frac{\partial z(x, y)}{\partial x} \right)^2 + \left( \frac{\partial z(x, y)}{\partial y} \right)^2 = k^2. \]  \hspace{1cm} (1.2)

Physicists refer to this partial differential equation as the eikonal equation. Expressing the boundary (base curve) in terms of the parameter \( t \), i.e., \( x = \hat{x}(t), \ y = \hat{y}(t) \), the equation (1.2) must also satisfy the boundary condition \( z(\hat{x}, \hat{y}) = 0 \) for the sand hill problem.

**Competition Problem:** Given an elliptical plate with semimajor axis \( a \) and semiminor axis \( b \) find the volume of sand \( V \) that this plate is capable of carrying in terms of the slope \( k \) for three different ratios of axes:

1) \( b / a = 0.25 \),  \hspace{1cm} 2) \( b / a = 0.5 \),  \hspace{1cm} 3) \( b / a = 0.75 \)

In the advent that an exact mathematical relationship cannot be found, alternative methods may be employed which give approximate values of volume for these three ratios.

Suggestions:

a) Find an exact analytical expression for the volume \( V \) using multiple integration techniques.
b) Find approximate expressions for volume by numerical integration for the three particular ratios of axes \( b/a \).
c) Make approximations to the elliptical boundary, which simplify the integration process.
d) Experimentally determine the volumes in terms of \( k \) using a granular material for the three particular ratios of axes by constructing cardboard bases of the appropriate
dimensions. Both volume and weight measurements may be required depending on the method employed. Some granular material such as salt will be provided for this approach together with various other materials and tools.

**Instructions:**

One team member should be chosen to fill out the last page of this handout (page 5), which is the official entry form for the competition. Include the name of your participating school and all team members as part of the entry. Detach the form from the introductory material before submitting the form to the competition judge.

Be sure to include as many significant figures (digits) in your answers as possible up to a maximum of sixteen for each ratio of b/a.

Contestants may use their own laptops and mathematics software as part of the competition. Reference books are also permitted. Limited computational facilities will be available on site to the contestants.

**Prize Criteria:**

The smallest total of absolute values of individual percent errors will be the winner. In the event of a tie the prize money will be shared equally among winning teams.

**Analytical Solution:**

\[
V = \frac{2}{3} ab^2 k (1+c) E \left( \frac{2\sqrt{c}}{1+c} \right), \quad \text{where} \quad c = \sqrt{1-\left(\frac{b}{a}\right)^2}
\]

\[
E(\delta) = \int_{0}^{\pi/2} \sqrt{1-\delta^2 \sin^2 t} \, dt
\]

Complete elliptic integral of the second kind
Official Entry Form for the MUPEC Design Competition  
April 26, 2003

Team School: ________________________________________________

Individual Team Members:  
1) ________________________________________________________
2) ________________________________________________________
3) ________________________________________________________
4) ________________________________________________________
5) ________________________________________________________
6) ________________________________________________________

Competition Problem: Given an elliptical plate with semimajor axis $a$ and semiminor axis $b$, find the maximum volume of granular material $V$ that the plate can support in terms of the slope $k$ of its surface for the following three ratios of axes:  
1) $b/a = 0.25$,  
2) $b/a = 0.5$,  
3) $b/a = 0.75$

Answers must be provided in the table below in the normalized (dimensionless) form indicated to as many significant figures (maximum sixteen) as possible:

| $b/a$ | $\frac{V}{kab^2}$ | $|\%\ Error|$ |
|-------|------------------|---------------|
| 0.25  | 1.313            |               |
| 0.50  | 1.255            |               |
| 0.75  | 1.165            |               |

$\Sigma |\%\ Error| =$ __________________________

*last column for official use only