PARAMETER IDENTIFICATION, MODELING, AND SIMULATION OF A CART AND PENDULUM

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ABSTRACT
In this paper a freely rotating pendulum suspended from a cart that slides on two parallel rails is modeled. In order to create this model several system parameters had to be experimentally determined. These parameters include the rotational damping and coulomb friction of the bearings attaching the pendulum to the cart, as well as the linear damping and coulomb friction of the linear bearings which the cart slides on. The results from a simple experiment were examined to determine each parameter. For the rotational terms the pendulum, suspended from a stationary cart, is given an initial displacement. The system is then modeled and the measured displacement is compared to the prediction of the model. For the translational terms the cart was modeled as free to oscillate between two springs in tension, the pendulum was omitted from this analysis. In each case, the simulations were performed using Dymola and Simulink computer packages and the difference between the experiment and the model was minimized by varying the parameter to be identified. Values were obtained for the rotational and linear damping, as well as the rotational and linear coulomb friction, which caused the model to agree nicely with the experiment.

Keywords: Modeling, Simulation, Cart and Pendulum, Simulink, Dymola

INTRODUCTION
For this project the cart and pendulum set-up in the Dynamic Systems and Controls Lab was utilized. The system itself is described in further detail below. The project goal is to create a model which accurately predicts both the pendulum position and the cart position. Although a model of the system existed previous to this project, it had not been updated in quite some time. Therefore it was desirable to redetermine several system parameters, as they appear to have changed over time. These parameters were determined by conducting two simple experiments and comparing the results to the experiment.

SYSTEM DESCRIPTION
The system consists of a slender rod attached by way of two bearings to a cart. The cart is mounted on four pillow block linear bearings, which slide along two parallel rails. The pendulum is free to swing below the cart. If the pendulum is mounted above the cart, the system is the conventional inverted pendulum system familiar in controls labs everywhere. The input to the system is the voltage supplied to a DC motor. The motor then produces a torque, which is put through a series of gears and then transmitted to the
cart through a plastic chain. User input through a computer controls the input voltage. A system schematic can be seen in Fig. 1.

**Descriptors**

The following variables are used to describe the behavior of the system.

- $\theta$ displacement angle of the pendulum
- $x$ displacement distance of the cart
- $\omega_m$ output velocity of the motor
- $\omega_b$ output velocity of the brake motor

**Measured Parameters**

The following parameters were measured from the physical system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_c$</td>
<td>4.609kg</td>
<td>mass of the cart</td>
</tr>
<tr>
<td>$m_p$</td>
<td>0.1719kg</td>
<td>mass of the pendulum</td>
</tr>
<tr>
<td>$l_p$</td>
<td>0.86m</td>
<td>length of the pendulum</td>
</tr>
<tr>
<td>$r_{g,1}$</td>
<td>0.0164m</td>
<td>radius of gear 1</td>
</tr>
<tr>
<td>$t_{g,1}$</td>
<td>0.00414m</td>
<td>thickness of gear 1</td>
</tr>
<tr>
<td>$r_{g,2}$</td>
<td>0.0252m</td>
<td>radius of gear 2</td>
</tr>
<tr>
<td>$t_{g,2}$</td>
<td>0.00419m</td>
<td>thickness of gear 2</td>
</tr>
<tr>
<td>$r_{hub}$</td>
<td>0.0376m</td>
<td>radius of hub</td>
</tr>
<tr>
<td>$t_{hub}$</td>
<td>0.0222m</td>
<td>thickness of hub</td>
</tr>
<tr>
<td>$r_{g,3}$</td>
<td>0.0181m</td>
<td>radius of gear 3</td>
</tr>
<tr>
<td>$t_{g,3}$</td>
<td>0.00945m</td>
<td>thickness of gear 3</td>
</tr>
</tbody>
</table>
Figure 1: System Schematic with Descriptors and Parameters Identified
Calculated Parameters

The system parameters given in Table 2 were calculated from the physical system using Eq. (1), (2), and (3). The mass moment of inertia of the pendulum was calculated using the measured mass and length of the pendulum and Eq. (1). It was not possible to physically remove the gears from the system in order to weigh them. It was therefore necessary to estimate the mass of each gear from measurable system quantities. The radius and thickness of each gear was measured, these data can be seen in Table 1. Gears 1 and 2 were assumed to be made of steel, while gear 3 was assumed to be made of aluminum, the specific weight of each substance is given below. The approximate mass of each gear was then computed using Eq. (2). Finally the mass moment of inertia of each gear could be computed using Eq. (3). The results of both of these computations can be seen in Table 2.

Mass moment of inertia of a slender rod—taken about the end point:

\[ I_o = \frac{1}{3} mL^2 \]  

(1)

Mass of a cylinder:

\[ m = \gamma \pi r^2 t \]  

(2)

where \( \gamma = 7860 \frac{kg}{m^3} \) for steel

and \( \gamma = 2767.97 \frac{kg}{m^3} \) for aluminum

Mass moment of inertia of a thin disk:

\[ I_G = \frac{1}{2} mr^2 \]  

(3)

Table 2: Calculated System Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{G,p} )</td>
<td>0.04238 ( \frac{N \cdot m}{rad/sec^2} )</td>
<td>mass moment of inertia of the pendulum</td>
</tr>
<tr>
<td>( m_{G,1} )</td>
<td>0.0275kg</td>
<td>mass of gear 1</td>
</tr>
<tr>
<td>( I_{G,1} )</td>
<td>3.698E-6 ( \frac{N \cdot m}{rad/sec^2} )</td>
<td>mass moment of inertia of gear 1</td>
</tr>
<tr>
<td>( m_{G,2} )</td>
<td>0.0657kg</td>
<td>mass of gear 2</td>
</tr>
<tr>
<td>( I_{G,2} )</td>
<td>2.086E-5 ( \frac{N \cdot m}{rad/sec^2} )</td>
<td>mass moment of inertia of gear 2</td>
</tr>
<tr>
<td>( m_{hub} )</td>
<td>0.775kg</td>
<td>mass of the hub</td>
</tr>
<tr>
<td>( I_{hub} )</td>
<td>5.478E-4 ( \frac{N \cdot m}{rad/sec^2} )</td>
<td>mass moment of inertia of the hub</td>
</tr>
<tr>
<td>( I_{sprocket} )</td>
<td>5.687E-4 ( \frac{N \cdot m}{rad/sec^2} )</td>
<td>mass moment of inertia of the sprocket</td>
</tr>
</tbody>
</table>
Provided

The following parameters were obtained from the manufacturer’s specifications web page [1].

**Motor**

- $m_{G,3} = 0.0269 \text{ kg}$ mass of gear 3
- $I_{G,3} = 4.406 \times 10^{-6} \frac{N \cdot m}{rad/\text{sec}^2}$ mass moment of inertia of gear 3

**Motor**

- $R_{a,m} = 0.739 \Omega$ armature resistance
- $L_{a,m} = 45 \mu H$ armature inductance
- $J_m = 1.20 \times 10^{-4} \frac{N \cdot m}{rad/\text{sec}^2}$ moment of inertia
- $c_m = 2.094 \times 10^{-6} \frac{N \cdot m}{rad/\text{sec}}$ viscous damping constant
- $K_{r,m} = 0.0178 \frac{V}{\text{RPM}}$ back emf constant
- $K_{i,m} = 0.1702 \frac{N \cdot m}{A}$ torque constant

**Brake Motor**

- $R_{a,b} = 1.120 \Omega$ armature resistance
- $L_{a,b} = 100 \mu H$ armature inductance
- $J_b = 0.19 \frac{N \cdot m}{rad/\text{sec}^2}$ moment of inertia
- $c_b = 7.330 \times 10^{-7} \frac{N \cdot m}{rad/\text{sec}}$ viscous damping constant
- $K_{r,b} = 0.00924 \frac{V}{\text{RPM}}$ back emf constant for the motor used as a brake
- $K_{i,b} = 0.0883 \frac{N \cdot m}{A}$ torque constant for the motor used as a brake
SYSTEM IDENTIFICATION

Rotational Damping

Pendulum Encoder Calibration

Experimental Set-Up
In order to determine the rotational damping of the bearings that attach the pendulum to the cart, a simple experiment was performed. The cart was held stationary, the pendulum was given an initial displacement from the horizontal, an encoder was used to measure $\theta$, and the data was acquired by a data acquisition system. A diagram of this experimental set-up can be seen in Fig. 2.

![Rotational Damping Experimental Set-Up](image)

Model Development
Initially the pendulum was modeled assuming only rotational damping was present. For this system Eq. (4) was derived as the equation of motion.

$$J\ddot{\theta} + b_r \dot{\theta} = -\frac{mg}{2}l \sin(\theta)$$

The best value for the damping will provide the smallest difference between the experimental data and the theoretical Dymola simulation results. To find this value a cost function, defined by Eq. (5), was used. This cost function is the sum-squared error between the experimental and theoretical data.

$$J = \sum (\theta_{\text{theoretical}} - \theta_{\text{experimental}})^2$$

The Dymola simulation was run in Simulink with several different values of $b_r$, and $J$ was calculated each time. A plot of several iterations of this process can be seen in Fig. 3. The damping constant which produced the lowest error value is

$$b_r = 0.0375 \frac{N \cdot m}{rad/s}.$$
After minimizing the error with only rotational damping, coulomb friction is added to the model. For this more complex system, Eq. (6) was derived as the equation of motion. In order to accurately model the coulomb friction, that is to make it always negative, the coulomb friction constant was multiplied by the sign of the angular velocity. In doing this, if the angular velocity is negative, a negative sign will be added to the coulomb friction term making it positive. However, if the angular velocity is positive, the coulomb friction term will remain negative.

\[
J \ddot{\theta} + b_r \dot{\theta} = - \frac{mg}{2} \sin(\theta) - c_r \text{sign}(\dot{\theta})
\]

(6)

The values of \( b_r \) and \( c_r \) that minimize the error between Eq. (6) and the experimental data are \( c_r = 0 \) and \( b_r = 0.0375 \frac{N \cdot m}{rad/s} \). Thus, the pendulum is most accurately modeled with rotational damping and no coulomb friction.

**Simulation Settings**

The experimental data was taken by an encoder with a sample frequency of 100 Hz. In order to easily calculate the error between the experimental data and the Simulink output it was desired that the Simulink model also run at a sample frequency of 100 Hz, that is a step size of 0.01 seconds. This was accomplished by making Simulink solve the model with a fixed time step size ODE solver, as opposed to a variable time step solver. For this particular simulation, a fourth order Runge-Kutta solver was used.
Simulation Results

After running the model several times a value of $b_r = 0.0375 \frac{N \cdot m}{rad/s}$ was found to minimize the error. A plot of the Simulink output and the experimental data for the final value of the rotational damping constant can be seen in Figure 2.

![Figure 4: Simulink and Experimental Data](image)

Linear Damping

Experimental Set-Up

In order to determine the linear damping of the cart, a spring was hooked to either end of the cart. The other end of each spring was then hooked into a clamp which was rigidly attached to the frame of the system. The cart was then deflected an initial distance to one side and released, while an encoder was used to measure the position of the cart. It was not possible to measure the position of the cart directly, but there was an encoder set-up to measure the output from a tachometer attached to the motor. For this reason the motor and the brake motor were left in the system, and the data from the tachometer was converted to give the position of the cart. A diagram of this experimental set-up can be seen in Fig. 5.
Spring Constant Determination

After the experiment was finished each spring was hung from a rigid support and various weights were hung on the free end. The deflection of the spring in centimeters was measured for each weight used. These values were then plotted in MatLab and the polyfit and polyval commands were used to determine a linear approximation of the data. These approximations can be seen below as Eq. (7) and (8). In each case the coefficient for the deflection term is the spring constant. Thus $K_1 = 1197.3N \cdot m$ and $K_2 = 1302.1N \cdot m$.

$$F_1 = 1197.3x_1 + 0.0014$$  \hspace{1cm} (7)

$$F_2 = 1302.1x_2 - 0.0001$$  \hspace{1cm} (8)

A plot of the experimental data and the linear approximation for each spring can be seen below in Figures 6 and 7.
Model Development

Initially the cart was modeled assuming that the moments of inertia of the gears were not negligible, and that the moments of inertia provided by the manufacturer for the motors were correct. The springs connecting the cart to the frame were assumed to be operating in their linear regions, and each spring was initially deflected 0.095m. This yielded Eq. (9) as the equation of motion.
\[ m_c + m_{\text{inertial}} \\cdot c_{\text{inertial}} + b_t \cdot K_1 (x + 0.095) - K_2 (0.095 - x) + c_e \cdot \text{sign} (\phi) = 0 \quad (9) \]

where \( m_{\text{inertial}} = \frac{\left( I_{g.1} + I_{g.2} \right)}{r_1} + \frac{\left( 2I_{g.3} + J_m + J_b \right)r_2}{r_1r_3} \)

and \( c_{\text{inertial}} = \frac{(c_m + c_b)r_2}{r_1r_3} \)

This simulation was run in Simulink and plotted against the experimental data, see Fig. 8.

![Figure 8: Initial Model for Linear Damping Terms](image)

Obviously Eq. (9) is not a very good model of the system. The most obvious reason for this discrepancy is that the moments of inertia for the motors are not correct. Another possibility is that the control system is acting in some way which is not understood at this time. A new model for the system was derived and can be seen in Eq. (10). This time the moments of inertia for the gears were assumed to be negligible, as they are very small and can be accounted for in the added inertial term. Also the coulomb friction was assumed to be 10N since that is force found in an informal experiment and it didn’t produce a noticeable effect on the simulation output. A spring scale was attached to the cart and used to pull the cart at a constant velocity, as this was done the scale read approximately 10N.
\[(m_r + m_{\text{inertial}}) - b_x - b_y - K_1(x + 0.095) - K_2(0.095 - x) + 10 = 0\]  
(10)

where \(m_{\text{inertial}}\) is an input being solved for

The best values for the damping and the additional mass of the system will provide the smallest difference between the experimental data and the theoretical Dymola simulation data. To find this value a cost function was used again. This cost function is the sum-squared error between the experimental data and theoretical results and can be seen in Eq. (5). First the linear damping term was changed while the inertial mass was held constant at 6.8. A plot of the various values for both the linear damping and the error can be seen in Fig. 9.

![Figure 9: Minimization of System Error with Inertial Mass Held Constant](image)

Next the inertial mass term was changed while the linear damping was held at 51. As above a plot of inertial mass and the system error can be seen in Fig. 10.

![Figure 10: Minimization of System Error with Linear Damping Held Constant](image)
Simulation Settings

The settings are the same as they were for the rotational damping simulation.

Simulation Results

After running the simulation several times final values of \( b_i = 48 \frac{N \cdot m}{rad/s} \) for the linear damping and \( m_{inertial} = 6.8kg \) for the added inertial mass of the system were found to minimize the error. A plot of the Simulink output and the experimental data for these values can be seen in Fig. 11.

![Figure 11: Final Model for Linear Damping Terms](image-url)
SUMMARY OF IDENTIFIED PARAMETERS

The following parameter values were experimentally determined.

Table 3: Experimentally Determined Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_r$</td>
<td>0.0375 Nm/srad</td>
<td>rotational damping of the pendulum bearings</td>
</tr>
<tr>
<td>$c_r$</td>
<td>0</td>
<td>coulomb friction of the pendulum bearings</td>
</tr>
<tr>
<td>$K_1$</td>
<td>1197.3 Nm</td>
<td>spring constant of spring 1</td>
</tr>
<tr>
<td>$K_2$</td>
<td>1302.1 Nm</td>
<td>spring constant of spring 2</td>
</tr>
<tr>
<td>$b_i$</td>
<td>48 Nm/srad</td>
<td>initial linear damping of the pendulum bearings</td>
</tr>
<tr>
<td>$m_{inertial}$</td>
<td>6.8 kg</td>
<td>final inertial mass of the system</td>
</tr>
<tr>
<td>$b_f$</td>
<td>51 Nm/srad</td>
<td>final linear damping of the pendulum bearings</td>
</tr>
</tbody>
</table>

CONCLUSIONS

Values were obtained for all of the parameters which were to be experimentally determined and can be seen in Table 3 above. The model for the rotational damping of pendulum bearings matched the experimental data with no additional changes to the model. With the linear model, however, it was found that inertial mass needed to be added to the system in order for the model and the experiment to agree. After this additional mass was included in the model, the simulation and the experiment matched quite well. This indicates that further work needs to be done in identifying the parameters of the system and understanding the control system.

FURTHER WORK

Experimentally identify the parameters of the motors
- inertial mass
- armature resistance
- armature inductance
- viscous damping constant

Construct and test a DAE model of the entire closed loop system
- controller in the loop
- the pendulum inverted

Construct and train a neural net model

ACKNOWLEDGEMENTS

Ben Howe for his help numerous times in setting up and running experiments.

REFERENCES

[1] www.motionvillage.com
APPENDIX A: SIMULINK MODEL AND ASSOCIATED DYMOLA CODE

Rotational Damping Model:

\[ \frac{b}{r} \]

\[ \frac{1}{c[r]} \]

\[ \theta \]

\[ t \]

\[ f \]

\[ \text{tvarg} \]

\[ \text{tvar} = \theta; \]
\[ \frac{d\theta(t)}{dt} = f; \]
\[ I\frac{df(t)}{dt} + b\cdot f + mg\cdot L/2 \cdot \sin(\theta) + c\cdot f = 0; \]

begin RotaDamp.mo
  class RotaDamp
    parameter Real m=0.1791;
    parameter Real L=0.86;
    parameter Real g=9.81;
    parameter Real I=0.0442;

    input Real b;
    input Real c;

    Real theta;

    Real f;
    output Real tvar;
    equation
      tvar = theta;
      der(theta) = f;
      I*der(f) + b*f + m*g*L/2*sin(theta) + c*f = 0;
  end RotaDamp
end RotaDamp.mo
In it is the Linear Damping Model:

\[ \text{begin LinearDamp.mo} \]

\text{class LinearDamp} \]

\text{parameter Real mc=4.609;} \]
\text{parameter Real Ig1=3.698E-6;} \]
\text{parameter Real Ig2=5.687E-4;} \]
\text{parameter Real Ig3=4.406E-6;} \]
\text{parameter Real Jb=0.19;} \]
\text{parameter Real Jm=1.2E-4;} \]
\text{parameter Real K1=1197.3;} \]
\text{parameter Real K2=1302.1;} \]
\text{parameter Real cb=7.330E-7;} \]
\text{parameter Real cm=2.094E-6;} \]
\text{parameter Real r1=0.0164;} \]
\text{parameter Real r2=0.0252;} \]
\text{parameter Real r3=0.0181;} \]

\text{Real tempx(start=0.1146);} \]
\text{Real TotalMass;} \]
\text{Real TotalDamp;} \]
\text{Real coulomb;} \]
\text{Real f;} \]
\text{input Real bL;} \]
\text{input Real cL;} \]
\text{output Real x(start=0.1146);} \]

\text{equation} \]
\text{TotalMass = mc + 2*(Ig1 + Ig2)/r1 + (2*Ig3 + Jm + Jb)*r2/(r1*r3);} \]
\text{TotalDamp = (cm + cb)*r2/(r1*r3) + bL;} \]
\text{tempx = x;} \]
\text{coul = cL*sign(f);} \]
\text{f = der(tempx);} \]

\text{TotalMass*der(f) + TotalDamp*f + K1*(x + 0.095) - K2*(0.095 - x) + cou = 0;} \]
\text{end LinearDampend LinearDamp.mo} \]
Modified Linear Damping Model:

\[
\text{begin LinearDampMod.mo}
\text{model LinearDampMod}
\text{parameter Real } mc=4.609;
\text{parameter Real } K1=1197.3;
\text{parameter Real } K2=1302.1;

\text{Real } \text{tempx};
\text{Real } f;

\text{input Real } \text{minert};
\text{input Real } bL;
\text{output Real } x;
\text{equation}
\text{tempx} = x;

f = \text{der(tempx)};

(mc + minert) \ast \text{der(f)} + bL \ast f + K1 \ast (x + 0.095) - K2 \ast (0.095 - x) + 10 = 0;
\text{end LinearDampMod}
\text{end LinearDampMod.mo}
APPENDIX B: MATLAB CODE

Rotational Damping Model:

begin RotaDampCode.m

% Import the output of the Simulink model (theta) and the time vector (tout) from
% the workspace
% Plot the Simulink output vs. experimental data
figure
plot(tangle,angle,tout,theta,'*')
axis([0.5,-0.4,0.3])
xlabel('Time (sec.)')
ylabel('Theta (rad.)')
grid
legend('Experimental Data','Theoretical Data')

% Calculate the squared error and plot it for several values of br
% Output the sum squared error for easy reference
Err2 = (angle - theta).^2;
J = sum(Err2)

% Vectors of the damping coefficient input to the simulink model
% and the output sum squared error
inputb = [0.01 0.02 0.03 0.035 0.037 0.0373 0.0375 0.0377 0.038 0.04
          0.041 0.042 0.05 0.06 0.07 0.1 0.15 0.2];
outputJ = [4.6219 1.5246 0.5125 0.3766 0.3632 0.3628 0.3629 0.3634
           0.3758 0.3872 0.4018 0.6017 0.9673 1.3741 2.5249 3.9557 4.9622];

figure
plot(inputb,outputJ,'*')
axis([0.034,0.043,0.35,0.41])
xlabel('damping coefficient')
ylabel('sum squared error')
end RotaDampCode.m

Linear Damping Model:

begin SpringConstants.m

% Calculation of Spring Constants
% the masses are in lbs the force is in Newtons

mass = [0 7.48 10.005 12.245 16.245];
force = mass.*0.454.*9.81;

% Spring 1
Length1 = [0.11 0.135 0.145 0.155 0.17];
Deflec1 = Length1-0.11;
[P1] = polyfit(Deflec1, force, 1)
y1 = polyval(P1, Deflec1);

% plot the experimental data points and the linear approximation
% for the deflection of spring 1
figure
plot(Deflec1,force,'*',Deflec1,y1)
grid
xlabel('Deflection (m)')
ylabel('Force (N)')
axis([0,0.06,0,80])

% Spring 2
Length2 = [0.125 0.15 0.16 0.1675 0.18];
Deflec2 = Length2-0.125;

[P2] = polyfit(Deflec2, force, 1)
y2 = polyval(P2, Deflec2);

% plot the experimental data points and the linear approximation
% for the deflection of spring 2
figure
plot(Deflec2,force,'*',Deflec2,y2)
grid
xlabel('Deflection (m)')
ylabel('Force (N)')
axis([0,0.06,0,80])
end SpringConstants.m

beginLinearDampCode.m
pos = exp.*0.0254;
time = [0:0.01:1.08];

% Import the output of the Simulink model (xout) and the time vector (tout) from
% the workspace plot the Simulink output vs. experimental data
plot(time,pos,'.',tout,xout)
xlabel('Time (sec.)')
ylabel('Position (m.)')
grid
legend('Experimental Data','Simulation Results')

% Calculate the squared error and plot it
% Output the sum squared error for easy reference
J = (pos - xout).^2;
sum(J)
endLinear DampCode.m